Advances in Mathematics 319 (2017) 111–152

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Faces of highest weight modules and the universal Weyl polyhedron

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A R T I C L E   I N F O

Article history:
Received 3 November 2016
Accepted 1 August 2017
Available online xxxx
Communicated by Ezra Miller

MSC:
primary 17B10
secondary 17B67, 22E47, 17B37, 20G42, 52B20

Keywords:
Highest weight module
Parabolic Verma module
Weyl polytope
Integrable Weyl group
f-polynomial
Universal Weyl polyhedron

A B S T R A C T

Let V be a highest weight module over a Kac–Moody algebra $\mathfrak{g}$, and let $\operatorname{conv} V$ denote the convex hull of its weights. We determine the combinatorial isomorphism type of $\operatorname{conv} V$, i.e. we completely classify the faces and their inclusions. In the special case where $\mathfrak{g}$ is semisimple, this brings closure to a question studied by Cellini and Marietti (2015) [7] for the adjoint representation, and by Khare (2016, 2017) [17, 18] for most modules. The determination of faces of finite-dimensional modules up to the Weyl group action and some of their inclusions also appears in previous works of Satake (1960) [25], Borel and Tits (1965) [3], Vinberg (1990) [26], and Casselman (1997) [6].

For any subset of the simple roots, we introduce a remarkable convex cone which we call the universal Weyl polyhedron, which controls the convex hulls of all modules parabolically induced from the corresponding Levi factor. Namely, the combinatorial isomorphism type of the cone stores the classification of faces for all such highest weight modules, as well as how faces degenerate as the highest weight gets increasingly singular. To our knowledge, this cone is new in finite and infinite type.

We further answer a question of Michel Brion, by showing that the localization of $\operatorname{conv} V$ along a face is always the convex hull of the weights of a parabolically induced module. Finally, as we determine the inclusion relations between faces representation-theoretically from the set of weights, without

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http://dx.doi.org/10.1016/j.aim.2017.08.005

recourse to convexity, we answer a similar question for highest weight modules over symmetrizable quantum groups.

Contents

1. Introduction

Throughout the paper, unless otherwise specified $\mathfrak{g}$ *is a Kac–Moody algebra over* $\mathbb{C}$ *with triangular decomposition* $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ *and Weyl group* W*, and* V *is a* $\mathfrak{g}$*-module of highest weight* $\lambda \in \mathfrak{h}^*$*.*

In this paper we study and answer several fundamental convexity-theoretic questions related to the weights of V and their convex hull, denoted $\operatorname{wt} V$ and $\operatorname{conv} V$ respectively. In contrast to previous approaches in the literature, our approach is to use techniques from representation theory to directly study $\operatorname{wt} V$ and deduce consequences for $\operatorname{conv} V$.

To describe our results and the necessary background, first suppose $\mathfrak{g}$ is of finite type and V is integrable. In this case, $\operatorname{conv} V$ is a convex polytope, called the Weyl polytope, and is well understood. In particular, the faces of such Weyl polytopes have been determined by Vinberg [26, §3] and Casselman [6, §3], and as Casselman notes, are implicit in the earlier work of Satake [25, §2.3] and Borel–Tits [3, §12.16]. The faces are constructed as follows. Let I index the simple roots of $\mathfrak{g}$, and $e_i, f_i, h_i,\ i \in I$ the Chevalley generators of $\mathfrak{g}$. Given $J \subset I$, denote by $\mathfrak{l}_J$ the corresponding Levi subalgebra generated by $e_j, f_j,\ \forall j \in J$ and $h_i,\ \forall i \in I$. Also define the following distinguished set of weights:

$$\operatorname{wt}_J V := \operatorname{wt} U(\mathfrak{l}_J) V_\lambda, \tag{1.1}$$

where V_λ is the highest weight space. The aforementioned authors showed that every face of the Weyl polytope for V is a Weyl group translate of the convex hull of $\operatorname{wt}_J V$ for some $J \subset I$.