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Full measure reducibility and localization for quasiperiodic Jacobi operators: A topological criterion

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We establish a topological criterion for connection between reducibility to constant rotations and dual localization, for the general family of analytic quasiperiodic Jacobi operators. As a corollary, we obtain the sharp arithmetic phase transition for the extended Harper's model in the positive Lyapunov exponent region.

1. Introduction

In this paper we study the general class of Jacobi operators

$$(H_c(\theta)u)_n = c(\theta + n\alpha)u_{n+1} + \tilde{c}(\theta + (n-1)\alpha)u_{n-1} + v(\theta + n\alpha)u_n, \tag{1.1}$$

where $c(\theta) = \sum_k \hat{c}_k e^{2\pi i k(\theta+\frac{\alpha}{2})} \in C^\omega(\mathbb{T})$, $\tilde{c}(\cdot) \in C^\omega(\mathbb{T})$, $\tilde{c}(\theta) = \overline{c(\theta)}$ on $\mathbb{T}$, and $v(\theta) = \sum_k \hat{v}_k e^{2\pi i k\theta} \in C^\omega(\mathbb{T})$. We will assume $\hat{v}_k = \overline{\hat{v}_{-k}}$, $\hat{c}_k \in \mathbb{R}$. Such operators arise as effective Hamiltonians in a tight-binding description of a crystal subject to a weak external magnetic field, with c, v reflecting the lattice geometry and the allowed electron

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hopping between lattice sites. The prime example, both in math and in physics literature, is the extended Harper's model, see (1.5). Notice that when $c(\theta) \equiv 1$ (this corresponds to the nearest neighbor hopping on a square lattice) we get the Schrödinger operator

$$(H(\theta)u)_n = u_{n+1} + u_{n-1} + v(\theta + n\alpha)u_n. \tag{1.2}$$

The Aubry dual of H_c is an operator $\tilde{H}_c$ defined by

$$(\tilde{H}_c(x)u)_m = \sum_{m'} d_{m'}(c,v)(x)u_{m-m'}, \tag{1.3}$$

where $d_{m'}(c,v)(x) = \hat{c}_{m'}e^{2\pi i(x-\frac{m'}{2}\alpha)} + \hat{v}_{-m'} + \hat{c}_{-m'}e^{-2\pi i(x-\frac{m'}{2}\alpha)}$.

The Aubry duality can be explained by the magnetic nature and corresponding gauge invariance of operators H_c [26] and has been formulated and explored on different levels, e.g. [26], [12], [5]. The dynamical formulation of Aubry duality is an observation that if $\tilde{H}_c(\theta)$ has an eigenvalue at E with respective eigenvector $\{u_n\}$, then, considering its Fourier transform, $u(x) := \sum_{n\in\mathbb{Z}} u_n \mathrm{e}^{2\pi inx} \in L^2(\mathbb{T}) \setminus \{0\}$ and letting

$$M_\theta(x) = \begin{pmatrix} u(x) & u(-x) \\ \mathrm{e}^{-2\pi i\theta}u(x-\alpha) & \mathrm{e}^{2\pi i\theta}u(-(x-\alpha)) \end{pmatrix}, \tag{1.4}$$

M_θ provides an L^2 semiconjugacy between the transfermatrix cocycle of H_c and the rotation $R_\theta = \begin{pmatrix} \mathrm{e}^{2\pi i\theta} & 0 \\ 0 & \mathrm{e}^{-2\pi i\theta} \end{pmatrix}$. For θ that are not α-rational, $\det M_\theta(x)$ doesn't vanish for a.e. x [6], leading to reducibility of the transfermatrix cocycle of H_c to a constant rotation R_θ. In particular, pure point spectrum for a.e. θ of $\tilde{H}_c(\theta)$ leads to reducibility for cocycles of H_c for a.e. E with respect to the density of states [27], [6], with the quality of reducibility governed by the rate of decay of u_n. As there are well developed methods to prove localization (thus exponential decay of the eigenfunctions) in various applications, this can be used to establish further interesting consequences [5,6,14].

With the development of recent powerful methods [7,4,2] to establish non-perturbative reducibility directly and independently of localization for the dual model, the reverse direction: obtaining localization for $\tilde{H}_c$ from reducibility of H_c, first used in a more restricted form back in [9], started gaining prominence. In the Schrödinger case, reducibility provides a direct construction of eigenfunctions for the dual model (with the decay governed by the quality of reducibility), so their completeness becomes the main issue. This has been considered a nontrivial question even for the almost Mathieu family. It had been conjectured for a long time [16] that $\lambda = e^\beta$, where β is the upper rate of exponential growth of denominators of continued fractions approximants to α (see (2.1)), is the phase transition line from purely singular continuous spectrum to pure point spectrum. A combination of the almost reducibility conjecture [2] and techniques of [4,15,29] led to establishing reducibility throughout the dual of the entire conjectured localization region, yet completeness of the resulting eigenfunctions remained a problem. This was