Advances in Mathematics 319 (2017) 251–291

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Convergence rate for spectral distribution of addition of random matrices

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ARTICLE INFO

Article history:
Received 16 July 2016
Received in revised form 16 July 2017
Accepted 15 August 2017
Available online 1 September 2017
Communicated by Dan Voiculescu

MSC:
46L54
60B10
60B20

Keywords:
Random matrices
Strong local law
Free convolution
Convergence rate

ABSTRACT

Let A and B be two N by N deterministic Hermitian matrices and let U be an N by N Haar distributed unitary matrix. It is well known that the spectral distribution of the sum $H = A + UBU^*$ converges weakly to the free additive convolution of the spectral distributions of A and B, as N tends to infinity. We establish the optimal convergence rate $\frac{1}{N}$ in the bulk of the spectrum.

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[1] Partially supported by ERC Advanced Grant RANMAT No. 338804 and Hong Kong RGC grant ECS 26301517.
[2] Partially supported by ERC Advanced Grant RANMAT No. 338804.
[3] Partially supported by ERC Advanced Grant RANMAT No. 338804 and the Göran Gustafsson Foundation.

http://dx.doi.org/10.1016/j.aim.2017.08.028

1. Introduction

In the influential work [21], Voiculescu showed that two independent large Hermitian matrices are asymptotically free if one of them is conjugated by a Haar distributed unitary matrix. This observation identifies the law of the sum of two large Hermitian matrices in a randomly chosen relative basis. More specifically, if $A = A^{(N)}$ and $B = B^{(N)}$ are two sequences of deterministic N by N Hermitian matrices and U is a Haar distributed unitary matrix, then the empirical eigenvalue distribution, μ_H, of the random sum $H = A+UBU^*$ is asymptotically given by the free additive convolution, $\mu_A \boxplus \mu_B$, of the eigenvalue distributions of A and B. A quantitative control of the closeness between μ_H and $\mu_A \boxplus \mu_B$, or the convergence rate of μ_H, has been out of reach until very recently. The first convergence rate $(\log N)^{-1/2}$ was obtained by Kargin in [15] by using the Gromov–Milman concentration inequality for the Haar measure. Later, Kargin improved in [16] his result to $N^{-1/7}$ in the bulk of the spectrum by studying the Green function subordination property down to the scale $N^{-1/7}$. Recently, we used in [1] a bootstrap argument to successively localize the Gromov–Milman inequality from larger to smaller scales, whereby we improved the convergence rate to $N^{-2/3}$.

In the current paper, we establish the convergence rate $N^{-1+\gamma}$, for any given $\gamma > 0$, in the bulk regime. Since the typical eigenvalue spacing in the bulk of the spectrum is N^{-1}, our result is optimal, up to the N^γ factor. In our recent work [2] on the local law of H we showed that the Green function subordination property holds down to the optimal scale $N^{-1+\gamma}$; *cf.* Proposition 3.2 below. In particular, the fluctuations of the matrix elements of the Green function $G(z) = (H - z)^{-1}$ were shown to be of order $N^{-1/2+\gamma}$ for any fixed z in the upper half plane, $\operatorname{Im} z > 0$. To get the optimal convergence rate, we need to show that the fluctuations of the normalized trace of the Green function, $\frac{1}{N} \operatorname{Tr} G$, are at most of order $N^{-1+\gamma}$. Thus the main task is to establish the fluctuation averaging of the diagonal entries of the Green function, *i.e.* that the fluctuations of the (weighted) average of the G_{ii}'s are typically as small as the square of the fluctuation of the G_{ii}'s; *cf.* (2.23).

Alongside with the convergence rate of μ_H to $\mu_A \boxplus \mu_B$, the concentration rate of μ_H to its expectation $\mathbb{E}\mu_H$ is of interest. An order $N^{-1/2}$ estimate up to logarithmic corrections on the fluctuations of the distribution function was obtained by Chatterjee in [9] by studying mixing times of random walks on the unitary group. Using the Gromov–Milman concentration inequality, Kargin removed the logarithmic corrections [15]. More recently, a rate of order $N^{-2/3}$ in the L^1-Wasserstein distance was obtained by E. Meckes and M. Meckes in [18]. From our main result it follows that μ_H, when restricted to bulk intervals, has concentration rate N^{-1}.

The fluctuation averaging mechanism is a key ingredient in proving the optimal convergence rate of local laws for random matrices. It was first introduced in [14] and substantially extended later in [12,13] to generalized Wigner matrices. In all previous works, however, the proofs heavily relied on the independence (up to symmetry) of the matrix elements. Our matrix $H = A+UBU^*$ lacks this independence since the columns