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The set of all orthogonal complex structures on the flat 6-tori

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ABSTRACT

In [2], Borisov, Salamon and Viaclovsky constructed non-standard orthogonal complex structures on flat tori $T^{2n}_{\mathbb{R}}$ for any $n \geq 3$. We will call these examples BSV-tori. In this note, we show that on a flat 6-torus, all the orthogonal complex structures are either the complex tori or the BSV-tori. This solves the classification problem for compact Hermitian manifolds with flat Riemannian connection in the case of complex dimension three.

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Contents

1. Introduction

Given a Hermitian manifold (M^n, g), there are several canonical metric connections on it that are well-studied. The Riemannian (or Levi-Civita) connection ∇ which is torsion free, and the Chern (aka Hermitian) connection ∇^c which is compatible with the complex structure, and the Bismut connection ∇^b, which is compatible with the almost complex structure and has skew-symmetric $(3, 0)$ torsion. When g is Kähler, all three connections coincide, but when g is not Kähler, the three are mutually distinct. Let us denote by R, R^c, and R^b the corresponding curvature tensors, respectively.

From the differential geometric point of view, it is very natural to study the curvature of each of these connections, and ask what kind of manifolds are "space forms" with respect to a given connection. In particular, one could ask what kind of compact complex manifolds will admit a Hermitian metric with flat Riemannian or Chern or Bismut connection?

For the Chern connection ∇^c, Boothby [1] proved in 1958 that compact Hermitian manifolds with $R^c = 0$ identically are exactly the compact quotients of complex Lie groups equipped with left invariant metrics. Such manifolds can be non-Kähler when $n \geq 3$. H.-C. Wang's complex parallelizable manifolds [16] form an important subset in this class.

For the Bismut connection ∇^b, in a recent work [17], we were able to show that compact Hermitian manifolds (M^n, g) with flat Bismut connections are exactly those covered by Samelson spaces, namely, $G \times \mathbb{R}^k$ equipped with a bi-invariant metric and a left invariant complex structure. Here G is a simply-connected compact semisimple Lie group, and $0 \leq k \leq 2n$. In particular, compact non-Kähler Bismut flat surfaces are exactly those isosceles Hopf surfaces, and in dimension three their universal cover is either a central Calabi–Eckmann threefold $S^3 \times S^3$, or $(\mathbb{C}^2 \setminus \{0\}) \times \mathbb{C}$. We refer the readers to [17] for more details. See also [6], [7], [8], [9], [12], [19] for expanded and/or related discussions or references.

So now we are left with the question of answering what kind of compact Hermitian manifolds (M^n, g) will have identically zero Riemannian curvature tensor? By Bieberbach Theorem, we know that such manifolds admit finite unbranched cover that is a flat torus