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Finite and infinite dimensional Lie group structures on Riordan groups

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ABSTRACT

We introduce a Frechet Lie group structure on the Riordan group. We give a description of the corresponding Lie algebra as a vector space of infinite lower triangular matrices. We describe a natural linear action induced on the Frechet space $\mathbb{K}^{\mathbb{N}}$ by any element in the Lie algebra. We relate this to a certain family of bivariate linear partial differential equations. We obtain the solutions of such equations using one-parameter groups in the Riordan group. We show how a particular semidirect product decomposition in the Riordan group is reflected in the Lie algebra. We study the stabilizer of a formal power series under the action induced by Riordan matrices. We get stabilizers in the fraction field $\mathbb{K}((x))$ using bi-infinite representations. We provide some examples. The main tool to get our results is the paper [18] where the Riordan group was described using inverse sequences of groups of finite matrices.

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1. Introduction

The Riordan group was introduced, under this name, by Louis Shapiro and collaborators in [29]. Since then, many authors have contributed to the development of this combinatorial device. Some of the contributions seek to clarify the algebraic structure of this group, many others focus on giving different ways to construct the elements of the Riordan group, which are called Riordan matrices or Riordan arrays. Many times the previous approaches allow to get recurrence relations for families of polynomials connecting to the study of classical sequences of polynomials. There are also many papers constructing new Riordan matrices from old ones. Another set of contributions try to find different frameworks where, we can say, the Riordan pattern appears, see [3–5,10, 22] as a sample. Overall, the Riordan group had been used as a machine to generate combinatorial identities thus giving new proofs to old ones or, directly, finding new ones. See [30] for a significant example. In [21], see also [17], a different approach was taken to describe the Riordan matrices. There, it was needed a complete metric on $\mathbb{K}[[x]]$ to get advantages of the procedure described by Banach's Fixed Point Theorem. This led to the authors to describe a complete ultrametric (or non-Archimedean metric) in the Riordan group converting it into a topological group.

Suppose that $\mathbb{K}$ is the field of real or complex numbers, denoted by $\mathbb{R}$ or $\mathbb{C}$ respectively. In this paper we are going to consider another natural way to give a completely metrizable topology in $\mathbb{K}[[x]]$ by means of the identification $\mathbb{K}^{\mathbb{N}} \equiv \mathbb{K}[[x]]$ obtained by passing from sequences to ordinary generating functions and viceversa. The topology considered in $\mathbb{K}^{\mathbb{N}}$ is always the product topology for the usual topology in $\mathbb{K}$. So, we convert $\mathbb{K}[[x]]$ into a Frechet space, that is, a completely metrizable locally convex vector topological space. This is the starting point to describe a natural Frechet Lie group structure on the Riordan group. Beside this, the Riordan group can be described as the inverse limit of an inverse sequence of finite dimensional matrix groups, see [18], obtaining so a pro-Lie group structure on the Riordan group $\mathcal{R}$ over the field $\mathbb{K}$, denoted by $\mathcal{R}(\mathbb{K})$. See [11] for an exhaustive topological treatment of pro-Lie groups.

Special comments are merited for the paper of R. Bacher [2] because some of the results obtained herein are first described there for the complex infinite dimensional case. Apparently, he did not use practically differential techniques, up to the beginning of the proof of Theorem 5.1 in [2] where he talks about *variable formelle*. There, he claims to give a proof of a matrix representation of the Lie algebra of the Riordan group. What Bacher seems to do there is to take the derivative of a curve in the Riordan group such that the topology induced by the implicit differential structure is the product topology whose factors are $\mathbb{K}$ countably infinite many times and $\mathbb{K}^*$ twice. What seems to be confusing is that, in the abstract, he justifies the existence of such differential structure by this way (this is taken literally from the English version of the abstract in [2]):

This group has a faithful representation into infinite lower triangular matrices and carries thus a natural structure as a Lie group.