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Sharp Hardy–Adams inequalities for bi-Laplacian on hyperbolic space of dimension four [☆]

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A B S T R A C T

We establish sharp Hardy–Adams inequalities on hyperbolic space $\mathbb{B}^4$ of dimension four. Namely, we will show that for any $\alpha > 0$ there exists a constant $C_\alpha > 0$ such that

$$\int_{\mathbb{B}^4} (e^{32\pi^2 u^2} - 1 - 32\pi^2 u^2) dV = 16 \int_{\mathbb{B}^4} \frac{e^{32\pi^2 u^2} - 1 - 32\pi^2 u^2}{(1-|x|^2)^4} dx \leq C_\alpha$$

for any $u \in C_0^\infty(\mathbb{B}^4)$ with

$$\int_{\mathbb{B}^4} \left(-\Delta_{\mathbb{H}} - \frac{9}{4}\right)(-\Delta_{\mathbb{H}} + \alpha) u \cdot u dV \leq 1.$$

As applications, we obtain a sharpened Adams inequality on hyperbolic space $\mathbb{B}^4$ and an inequality which improves the classical Adams' inequality and the Hardy inequality simultaneously. The later inequality is in the spirit of the Hardy–Trudinger–Moser inequality on a disk in dimension two given by Wang and Ye in [46] and on any convex planar domain by Lu and Yang in [33].

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The Fourier analysis techniques on hyperbolic and symmetric spaces play an important role in our work.

1. Introduction

Our main purpose of this article is to establish sharp Hardy–Adams inequalities on $\mathbb{B}^4$, a hyperbolic space in dimension four.

We first recall the classical Trudinger–Moser inequality in any finite domain of Euclidean spaces. Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded domain and $1 \leq q \leq \frac{np}{n-kp}$. Then it is well known that the Sobolev embedding theorem tells us the embedding $W_0^{k,p}(\Omega) \subset L^q(\Omega)$ is continuous when $kp < n$. However, in general $W_0^{1,n}(\Omega) \nsubseteq L^\infty(\Omega)$. Trudinger [45] established in the borderline case that $W_0^{1,n}(\Omega) \subset L_{\varphi_n}(\Omega)$, where $L_{\varphi_n}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_n(t) = \exp(\beta|t|^{n/n-1}) - 1$ for some $\beta > 0$ (see also Yudovich [49], Pohozaev [43]). In 1971, Moser sharpened the Trudinger inequality in [39] by finding the optimal β:

Theorem 1.1 *(Trudinger–Moser). Let Ω be a domain with finite measure in Euclidean n-space $\mathbb{R}^n$, $n \geq 2$. Then there exists a sharp constant $\beta_n = n\left(\frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\right)^{\frac{1}{n-1}}$ such that*

$$\frac{1}{|\Omega|}\int\limits_{\Omega} \exp(\beta|u|^{\frac{n}{n-1}})dx \leq c_0$$

for any $\beta \leq \beta_n$, any $u \in W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u|^n dx \leq 1$. This constant β_n is sharp in the sense that if $\beta > \beta_n$, then the above inequality can no longer hold with some c_0 independent of u.

In 1988, D. Adams extended such an inequality to higher order Sobolev spaces. In fact, Adams proved the following:

Theorem 1.2. *Let Ω be a domain in $\mathbb{R}^n$ with finite n-measure and m be a positive integer less than n. There is a constant $c_0 = c_0(m,n)$ such that for all $u \in C^m(\mathbb{R}^n)$ with support contained in Ω and $\|\nabla^m u\|_{n/m} \leq 1$, the following uniform inequality holds*

$$\frac{1}{|\Omega|}\int\limits_{\Omega} \exp(\beta_0(m,n)|u|^{n/(n-m)})dx \leq c_0, \tag{1.1}$$

where