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Extensions in Jacobian algebras and cluster categories of marked surfaces ☆,☆☆

Ilke Canakci [a], Sibylle Schroll [b,*]

[a] Department of Mathematical Sciences, Durham University, Stockton Road, Durham DH1 3LE, United Kingdom

[b] Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, United Kingdom

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ABSTRACT

In the context of representation theory of finite dimensional algebras, string algebras have been extensively studied and most aspects of their representation theory are well-understood. One exception to this is the classification of extensions between indecomposable modules. In this paper we explicitly describe such extensions for a class of string algebras, namely gentle algebras associated to surface triangulations. These algebras arise as Jacobian algebras of unpunctured surfaces. We relate the extension spaces of indecomposable modules to crossings of generalised arcs in the surface and give explicit bases of the extension spaces for indecomposable modules in almost all cases. We show that the dimensions of these extension spaces are given in terms of crossing arcs in the surface.

Our approach is new and consists of interpreting snake graphs as indecomposable modules. In order to show that our basis is a spanning set, we need to work in the associated cluster category where we explicitly calculate the middle terms of extensions and give bases of their extension spaces. We note

☆ With an appendix by Claire Amiot. Institut Fourier-UMR 5582, 100 rue des maths, 38402 Saint Martin d'Héres, France, *e-mail address:* claire.amiot@univ-grenoble-alpes.fr.

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* Corresponding author.
E-mail addresses: IlkeCanakci@lgmail.com (I. Canakci), schroll@le.ac.uk (S. Schroll).

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that not all extensions in the cluster category give rise to extensions for the Jacobian algebra.

1. Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in 2002 in [18] in order to give an algebraic framework for the study of the (dual) canonical bases in Lie theory. This work was further developed in [3,19,20]. Cluster algebras are commutative algebras given by generators, the *cluster variables*, and relations. The construction of the generators is a recursive process from an initial set of data. In general, even in small cases, this is a complex process. However, there is a class of cluster algebras coming from surfaces [16,17] (see also [14,15]) where this process is encoded in the combinatorial geometry of surface triangulations. Surface cluster algebras are an important part of the classification of (skew-symmetric) cluster algebras in terms of mutation type, namely almost all cluster algebras of finite mutation type are surface cluster algebras [13].

Cluster algebras from surfaces have been widely studied via the combinatorial geometry of the corresponding surfaces [13,16,17,32,33]. The same holds true for the associated cluster categories and Jacobian algebras. An important example of this is the crossing of two arcs in a surface. In the case of cluster algebras this gives rise to a multiplication formula for the corresponding cluster variables [34]. In the cluster category, the number of crossings of two arcs gives the dimension of the extension space between the associated indecomposable objects [42,38]. For Jacobian algebras of surfaces where all marked points lie in the boundary, in [6], building on [5] and [2], Auslander–Reiten sequences have been given in terms of arcs in the surface. In general, however, there has so far been no link between arbitrary crossings of arcs in the surface and the extensions between indecomposable modules in the Jacobian algebra.

The Jacobian algebras under consideration are gentle algebras and their indecomposable modules, given by strings and bands, correspond to curves and closed curves in the surface [2] (see [26,27,10,28] for a more general definition of Jacobian algebras via quiver with potential and [21] for classification of their representation type). Gentle algebras form a special class of algebras, for example, this class is closed under tilting and derived equivalence [40] and [41]. They are part of the larger family of string algebras which are an important family of algebras of tame representation type whose representation theory is well-understood. For example, their Auslander–Reiten structure has been determined [5] and in [11,25] the morphisms between indecomposable modules are completely described. However, a complete description of the extensions between indecomposable modules is not known.

In the present paper, we describe extension spaces of string modules over gentle Jacobian algebras. Furthermore, we show that in analogy with the cluster category, in most