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An infinite self-dual Ramsey theorem ☆

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A R T I C L E I N F O

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A B S T R A C T

In a recent paper [5] S. Solecki proved a finite self-dual Ramsey theorem that extends simultaneously the classical finite Ramsey theorem [4] and the Graham–Rothschild theorem [2]. In this paper we prove the corresponding infinite dimensional version of the self-dual theorem. As a consequence, we extend the classical infinite Ramsey theorem [4] and the Carlson–Simpson theorem [1].

1. Introduction

Recall that the classical infinite version of Ramsey's theorem [4] states that given any finite coloring of the set of all K element subsets of ω there exists an infinite subset A of ω such that the restriction of the coloring on all subsets of A of cardinality K is constant.

The dual form of Ramsey's theorem, the Carlson–Simpson theorem [1], states that given any finite Borel coloring of the set of all K-partitions of ω into K many classes,

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there exists a partition r of ω into ω many classes such that the set of all K partitions of ω which are coarser than r is monochromatic.

As it is well known the above results have finite versions as well, namely the finite Ramsey theorem and the Graham–Rothschild theorem [2]. Recently S. Solecki proved [5] a self-dual theorem which extends simultaneously the finite version of the Ramsey theorem and the Graham–Rothschild theorem.

The main goal of the present paper is to obtain an infinite self-dual Ramsey theorem which also extends simultaneously the infinite Ramsey theorem [4] and the Carlson–Simpson theorem [1].

We explicit state our contribution in Section 2. The rest of this introduction is devoted to the presentation of some background material. We also include a brief review of related work.

1.1. Basic definitions

We follow the terminology introduced in [5]. Let $K \leq L \leq \omega$. We view a natural number K as a linear order $(K, \leq_K)$ where $\leq_K$ is the usual ordering on the set $\{0, \dots, K-1\}$. Similarly in the case $K = \omega$. By a rigid surjection $t : L \to K$ we mean a surjection with the additional property that images of initial segments of L are also initial segments of K. We denote the image of a rigid surjection t, by $im(t)$ and its domain by $dom(t)$. Now let $t : L \to K$ and $i : K \to L$ be two maps. We say that the pair (t, i) is a connection if for all $x \in K, y \in L$:

$$t(i(x)) = x \text{ and if } y \leq i(x) \text{ then } t(y) \leq x.$$

Note that if (t, i) is a connection, then t is a rigid surjection and i is an increasing injection.

With this terminology the classical Ramsey Theorem can be stated as follows. Let l, K be natural numbers. For any l-coloring of all increasing injections $j : K \to \omega$ there exists an increasing injection $j_0 : \omega \to \omega$ such that the set $\{ j_0 \circ j : j : K \to \omega \}$ is monochromatic.

Similarly, the Carlson–Simpson Theorem can be stated as follows. Let l a natural number. For any Borel l-coloring of all rigid surjections $s : \omega \to K$, there exists a rigid surjection $s_0 : \omega \to \omega$ such that the set $\{ s \circ s_0 : s : \omega \to K \}$ is monochromatic.

1.2. The space of connections

Given a finite, possibly empty, alphabet $A = \{ \alpha_0, \dots, \alpha_{|A|-1} \}$, written in an increasing order, and given $K \leq L \leq \omega$ we define