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Quantum integrability and generalised quantum Schubert calculus

Vassily Gorbounov [a], Christian Korff [b,*]

[a] *Institute of Mathematics, University of Aberdeen, Aberdeen AB24 3UE, UK*
[b] *School of Mathematics and Statistics, University of Glasgow, Glasgow G12 8QQ, UK*

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ABSTRACT

We introduce and study a new mathematical structure in the generalised (quantum) cohomology theory for Grassmannians. Namely, we relate the Schubert calculus to a quantum integrable system known in the physics literature as the asymmetric six-vertex model. Our approach offers a new perspective on already established and well-studied special cases, for example equivariant K-theory, and in addition allows us to formulate a conjecture on the so-far unknown case of quantum equivariant K-theory.

* Corresponding author.
E-mail addresses: v.gorbounov@abdn.ac.uk (V. Gorbounov), christian.korff@glasgow.ac.uk (C. Korff).

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1. Introduction

Generalised complex oriented cohomology first appeared in the work of Novikov [55] and Quillen [59] who realised that formal groups naturally enter in algebraic topology. Such a theory is known to be completely characterised by the isomorphism $h^*(\mathbb{C}P^\infty) \cong h^*(\text{pt})[x]$, where x is the first Chern class of the canonical line bundle over the infinite complex projective space $\mathbb{C}P^\infty$, and the Künneth formula, $h^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong h^*(\text{pt})[x, y]$, which implies that the first Chern class of the tensor product of two line bundles obeys a formal group law [1]. There are three known types of formal group laws which come from the one-dimensional connected algebraic groups, the additive group, the multiplicative group, and elliptic curves, describing respectively (ordinary) cohomology, K-theory and elliptic cohomology.

On the other hand to each of the mentioned groups one can associate rational, trigonometric and elliptic solutions of the Yang–Baxter equation which are linked to the appropriate quantum groups. It was first suggested in [21] that there should be a connection between the latter and the mentioned generalised cohomology theories.

The study of solutions of the Yang–Baxter equation is at the heart of the area of quantum integrable systems. Based on earlier pioneering works of Hans Bethe [8] and Rodney Baxter [4], the Faddeev School [18] developed the *algebraic Bethe ansatz* or *quantum inverse scattering method*, where starting from a solution of the Yang–Baxter equation one constructs the quantum integrals of motion of the physical system as a commutative subalgebra, now often called the *Bethe algebra*, within a larger non-commutative *Yang–Baxter algebra*. Historically, Yang–Baxter algebras were the origin for the later definition of quantum groups by Drinfeld [17] and Jimbo [32]. Using the commutation relations of the Yang–Baxter algebra the Bethe ansatz culminates in the derivation of a set of – in our setting – polynomial equations, whose solutions describe the spectrum of the commuting transfer matrices which generate the Bethe algebra. In general solving these equations analytically is regarded as an intractable problem within the integrable systems community except for a few special cases.

The use of quantum integrability in the study of quantum cohomology of full flag varieties and quantum K-theory goes back to works of Givental, Kim and Givental, Lee; see [22,24,33,34] and [23,42]. In more recent work of Nekrasov and Shatashvili [54], which was further developed mathematically by Braverman, Maulik and Okounkov [9,47], it was established that the Bethe ansatz equations of some well known quantum integrable systems related to the quantum groups known as Yangians describe the quantum cohomology and quantum K-theory for a large class of algebraic varieties, the Nakajima varieties. Particular examples are the cotangent spaces of partial flag varieties, see the work [27], the simplest case being the cotangent space of the Grassmannian. This opens up an exciting new perspective on the connection made in [21].

In this article we shall instead investigate the above connection for the Grassmannians $\text{Gr}_{n,N} = \text{Gr}_n(\mathbb{C}^N)$ themselves rather than their cotangent spaces based on the earlier findings in [38,37,26]; see also the work on non-quantum $GL(N)$-equivariant cohomology