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On hyperbolic graphs induced by iterated function systems ☆

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ABSTRACT

For any contractive iterated function system (IFS, including the Moran systems), we show that there is a natural hyperbolic graph on the symbolic space, which yields the Hölder equivalence of the hyperbolic boundary and the invariant set of the IFS. This completes the previous studies ([16,20,30]) by eliminating superfluous conditions, and admits more classes of sets (e.g., the Moran sets). We also show that the bounded degree property of the graph can be used to characterize certain separation properties of the IFS (open set condition, weak separation condition); the bounded degree property is particularly important when we consider random walks on such graphs. This application and the other application to Lipschitz equivalence of self-similar sets will be discussed.

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1. Introduction

Let $\{S_j\}_{j=1}^N$ be a contractive iterated function system (IFS) on $\mathbb{R}^d$, and let K be the invariant set (attractor) generated by the IFS. It is well-known that the IFS is associated to a finite word space (symbolic space or coding space) Σ^*, which is equipped naturally with a tree structure and a visual metric. The limit set Σ^∞ of the tree is a Cantor set (topological boundary). Each element of K has a symbolic representation in Σ^∞, i.e., there is a canonical surjection $\tau : \Sigma^\infty \to K$, and K is homeomorphic to the quotient space $\Sigma^\infty / \sim$, where the equivalence relation is defined by $\tau(x) = \tau(y)$. In general one would like to impose more information on Σ^* so as to carry out further analysis on K. With the intention to bring in the probabilistic potential theory to K, Denker and Sato [6–8] first constructed a special type of Markov chain $\{Z_n\}_{n=0}^\infty$ on Σ^* of the Sierpinski gasket (SG), and showed that the Martin boundary of $\{Z_n\}_{n=0}^\infty$ is homeomorphic to the SG. Motivated by this, Kaimanovich [16] introduced the concept of "augmented tree" on Σ^* by adding new edges to the tree Σ^* according to the intersection of the cells of the IFS, he showed that the graph of the SG is hyperbolic in the sense of Gromov ([14, 31]), and that the SG is Hölder equivalent to the hyperbolic boundary of the augmented tree. He also suggested that this approach might also work for other IFS, and the device can be useful to bring in considerations on geometric groups into the study of fractal sets.

The above initiations were carried out by the authors in a series of papers ([3,15,20, 21,30]). In [20], we showed that the hyperbolic boundary and the self-similar set K are Hölder equivalent provided that the IFS satisfies the open set condition (OSC) together with a technical "condition (H)" on K (see Section 2). The Hölder equivalence was used to study the Lipschitz classification of the totally disconnected self-similar sets ([5,24]), and more generally the Moran sets [23].

In this paper, we unify the previous approaches and obtain the full generality of the Hölder equivalence of the hyperbolic boundaries and the attractors for the general contractive IFS's. We define the augmented tree on a tree with an associated set-valued map; we also relax the set of augmented edges used previously, so as to remove the OSC on the IFS and the condition (H) on the attractors.

Let X be an infinite set, and let $(X, \mathcal{E})$ be a locally finite connected tree. We fixed a reference point $o \in X$ as a *root* of the tree. For a vertex $x \in X$, we use $|x|$ to denote the length of a non-self-intersecting path from the root to x, and let $X_n = \{x : |x| = n\}$. Let $\Sigma(x) = \{y \in X : \ |y| = |x| + 1, \ (x, y) \in \mathcal{E}\}$ be the set of *offsprings* of x.

For our purpose, we will denote the edge set of the tree by $\mathcal{E}_v$, the set of *vertical* edges. Let $\mathcal{K}$ be the collection of nonempty compact subsets of $\mathbb{R}^d$. We associate with the tree $(X, \mathcal{E}_v)$ a set-valued map $\Phi : X \to \mathcal{K}$ satisfying

(A1) $\Phi(y) \subset \Phi(x)$ for all $y \in \Sigma(x)$;

(A2) $\exists\ \delta_0 > 1,\ 0 < r < 1 \ni \delta_0^{-1} r^n \le |\Phi(x)| \le \delta_0 r^n$ for all $x \in X_n$, where $|E|$ denotes the diameter of E.