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# A construction scheme for non-separable structures ☆

Stevo Todorcevic[a,b,*]

[a] *Department of Mathematics, University of Toronto, Canada*
[b] *Institut de Mathématiques de Jussieu, Paris, France*

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ABSTRACT

We describe a simple and general construction scheme for describing mathematical structures on domain $\omega_1$. Natural requirements on this scheme will reduce the nonseparable structural properties of the resulting mathematical object to some finite-dimensional problems that are easy to state and frequently also easy to solve. To illustrate usefulness of this scheme we give some applications mainly towards compact convex sets and normed spaces.



## 1. Introduction

In this paper we propose a construction scheme for describing a rich array of mathematical structures with domain $\omega_1$. As it will be seen the scheme is quite simple and easy to apply and with some extra natural requirement will reduce the control over the nonseparable structural properties of the resulting mathematical object to some easily stated finite-dimensional problems. Before we go any further, let us describe the historical setting where this scheme can naturally be placed.

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* Correspondence to: Department of Mathematics, University of Toronto, Canada.
*E-mail addresses:* stevo@math.toronto.edu, todorcevic@math.jussieu.fr.

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In the contexts of set theory and model theory there is a quite long and rich history of attempts to organize constructions of uncountable mathematical structures. Not much of this is lost even if we concentrate on construction of mathematical structures with domain equal to the first uncountable ordinal $\omega_1$. The first important constructions on this domain whose ideas were present in all other constructions since then are the construction of a Hausdorff gap in the quotient algebra $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ ([5,6]) and the construction of an Aronszajn tree ([9,10]). A particularly important feature that one finds in both of these constructions is the idea that the recursive construction of a structure on $\omega_1$ is done in such a way that for many (or all) $\delta < \omega_1$, the structure on $\delta$ is an elementary substructure of the whole structure, i.e., it reflects all the properties of the final structure describable with parameters $< \delta$. Later this was correctly generalized by the Keisler completeness theorem for the logic $L(Q)$, where $Q$ is the quantifier 'there exist uncountably many' (see [8]). Thus, the existence of a structure on an $\omega_1$-like domain is reduced to the consistency of some theory. Unfortunately, Keisler's theorem is not always easy to apply and this is frequently related to the fact that passing from a structure on an $\omega_1$-*like* domain to a structure with domain $\omega_1$ is not always evident. It is for this reason that in the literature one finds constructions that could be subjects of Keisler's theorem but are done directly using of course the same ideas (see, for example [17,21]).

In [14], Magidor and Malitz generalized Keisler theorem to logics $L(Q^n)$, where $Q^n$ is the higher dimensional analog of $Q$. This generalization involves Jensen's principle $\diamondsuit$, since the structures which their completeness theorem gives (like, for example, Souslin tree) cannot in general be constructed on the basis the ordinary (ZFC) axioms of set theory. Their completeness theorem has similar disadvantages as Keisler's theorem and for this reason one finds in the literature results that could be subject to their theorem but are done by direct constructions on $\omega_1$ using $\diamondsuit$ (see, for example, [18]).

Another way to organize the constructions of uncountable structures that belongs to the same general setting is Jensen's scheme of gap-1 morasses (see [3]) that has been greatly simplified by Velleman [22]. Here the goal is to construct a structure $\mathcal{M}$ with a unary predicate $P$ where the cardinality of the domain of $\mathcal{M}$ is equal to the successor of the cardinality of the interpretation of $P$. In the case when $\mathcal{M}$ has cardinality $\aleph_1$ this reduces to Keisler's theorem so it is not surprising that Velleman's version of Jensen's gap-1 scheme for $\omega_1$ is a consequence of the Keisler completeness theorem for $L(Q)$ (see the corresponding comment in [21]).

The scheme that we describe in this paper takes ideas from all of these sources but it also adds a crucial new idea inspired by some forcing constructions found in [1] and in [12]. What these forcing constructions reveal is that the control of the nonseparable structural properties of the resulting mathematical object on the domain $\omega_1$ is controlled by amalgamation of many isomorphic finite substructures. The amalgamation requirement is of course present in all of the previous attempts discussed above but all of them talk about amalgamations of *two* structures. To illustrate this, we give several examples in contexts that require little or no prerequisites. However, we believe that the full potential of our construction scheme could only be achieved with experts in different