Advances in Mathematics 313 (2017) 628–688

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Dirac operators in tensor categories and the motive of quaternionic modular forms

Marc Masdeu [a,*], Marco Adamo Seveso [b]

[a] *Department of Mathematics, University of Warwick, Coventry, United Kingdom*
[b] *Dipartimento di Matematica "Federigo Enriques", Università di Milano, Via Cesare Saldini 50, 20133 Milano, Italy*

A R T I C L E   I N F O

Article history:
Received 6 February 2015
Received in revised form 22 March 2017
Accepted 30 March 2017
Communicated by Ravi Vakil

Keywords:
Chow motive
Dirac operator
Quaternionic modular forms

A B S T R A C T

We define a motive whose realizations afford modular forms (of arbitrary weight) on an indefinite division quaternion algebra. This generalizes work of Iovita–Spiess to odd weights in the spirit of Jordan–Livné. It also generalizes a construction of Scholl to indefinite division quaternion algebras, and provides the first motivic construction of new-subspaces of modular forms.

Contents

* Corresponding author.
E-mail addresses: masdeu@mat.uab.cat (M. Masdeu), seveso.marco@gmail.com (M.A. Seveso).

http://dx.doi.org/10.1016/j.aim.2017.03.034
0001-8708/

1. Introduction

The paper [22] offers the construction of a motive whose realizations afford modular forms of even or odd weight on the indefinite split quaternion algebra over $\mathbb{Q}$. In [12, §10] the authors construct a motive of even weight modular forms on a quaternion division algebra (see also [24]). Based on ideas of Jordan and Livné (see [13]), this motive is constructed as the kernel of an appropriate Laplace operator. More precisely, let $h(A)$ be the motive of an abelian scheme A of relative dimension d over a smooth base scheme S (see [4] and [16]). It decomposes as the direct sum

$$h(A) = h^0(A) \oplus h^1(A) \oplus \ldots \oplus h^g(A)$$

where $g = 2d$ and there are canonical identifications

$$h^i(A) = \vee^i h^1(A),\ h^i(A) \simeq h^{2d-i}(A)^\vee(-d) \text{ and } h^{2d}(A) \simeq \mathbb{I}(-d),$$

where $\vee^\cdot V$ denotes the symmetric algebra of the object V. It follows that the multiplication morphisms

$$\varphi_{i,2d-i} : \vee^i h^1(A) \otimes \vee^{2d-i} h^1(A) \to \mathbb{I}(-d)$$

are perfect. In particular, taking $i = d$, one gets an associated Laplace operator[1]

$$\Delta^n : \mathrm{Sym}^n\left(\vee^d h^1(A)\right) \to \mathrm{Sym}^{n-2}\left(\vee^d h^1(A)\right)(-d),\ n \geq 2$$

and it is possible to show that the kernel exists. The following remark has been employed in [12, §10]. When A is an abelian scheme of dimension $d = 2$ with multiplication by the quaternion algebra B, we have that $B \otimes B$ acts on $\vee^2 h^1(A)$ and there is a canonical direct sum decomposition

$$\vee^2 h^1(A) = \left(\vee^2 h^1(A)\right)_+ \oplus \left(\vee^2 h^1(A)\right)_-$$

is such a way that $B^\times \subset B \otimes B$ (diagonally) acts via the reduced norm on $\left(\vee^2 h^1(A)\right)_-$. Furthermore, since the idempotents giving rise to this decomposition are self-adjoint with respect to $\varphi_{2,2}$, it follows that the induced pairing

$$\left(\vee^2 h^1(A)\right)_- \otimes \left(\vee^2 h^1(A)\right)_- \hookrightarrow \vee^2 h^1(A) \otimes \vee^2 h^1(A) \to \mathbb{I}(-2)$$

[1] For a symmetric or alternating power M we will write $\mathrm{Sym}^n(M)$ and $\mathrm{Alt}^n(M)$ when considering its symmetric or alternating powers once again.