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The integer cohomology algebra of toric arrangements

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A R T I C L E   I N F O

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A B S T R A C T

We compute the cohomology ring of the complement of a toric arrangement with integer coefficients and investigate its dependency from the arrangement's combinatorial data. To this end, we study a morphism of spectral sequences associated to certain combinatorially defined subcomplexes of the toric Salvetti category in the complexified case, and use a technical argument in order to extend the results to full generality. As a byproduct we obtain:

– a "combinatorial" version of Brieskorn's lemma in terms of Salvetti complexes of complexified arrangements,
– a uniqueness result for realizations of arithmetic matroids with at least one basis of multiplicity 1.

Contents

1. Introduction 747
2. Overview and statement of results 749
3. Preparations 756
4. Combinatorial topology of toric arrangements 765

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5. The complexified case . . . 778
6. The general case . . . 788
7. Dependency on the poset of layers . . . 794
Acknowledgments . . . 801
References . . . 801

1. Introduction

The goal of this paper is to give a presentation of the cohomology ring with integer coefficients of the complement of a toric arrangement – i.e., of a family of level sets of characters of the complex torus – and to investigate its dependency from the poset of layers of the arrangement.

This line of research can be traced back to Deligne's seminal work on complements of normal crossing divisors in smooth projective varieties [13] and has been extensively and successfully carried out in the case of arrangements of hyperplanes in complex space, where the integer cohomology ring of the complement is a well-studied object with strong combinatorial structure. In particular, it can be defined purely in terms of the intersection poset of the arrangement, and in greater generality, for any matroid, giving rise to the class of so-called *Orlik–Solomon algebras*. We refer to Yuzvinsky's survey [38] for a thorough introduction and a "tour d'horizon" of the range of directions of study focusing on OS-algebras.

Recently, the study of hyperplane arrangements has been taken as a stepping stone towards different kinds of generalizations. Among these let us mention the work of Dupont [16] developing algebraic models for complements of divisors with hyperplane-like crossings and of Bibby [2] studying the rational cohomology of complements of arrangements in abelian varieties. Both apply indeed to the case of interest to us, that of toric arrangements.

Besides being a natural step beyond arrangements of hyperplanes in the study of complements of divisors, our motivation for considering toric arrangements stems also from recent work of De Concini, Procesi and Vergne which puts topological and combinatorial properties of toric arrangements in a much wider context (see [12] or the book [11]) and spurred a considerable amount of research aimed at establishing a suitable combinatorial framework. This research was tackled along two main directions.

One such direction, from algebraic combinatorics, led Moci [25] to introduce a suitable generalization of the Tutte polynomials and then, jointly with d'Adderio [7], to the development of arithmetic matroids (for an up-to date account see Brändén and Moci [4]). These objects, as well as others like matroids over rings [18], exhibit an interesting structure theory and recover earlier enumerative results by Ehrenborg, Readdy and Slone [17] and Lawrence [20] but, as of yet, only bear an enumerative relationship with topological or geometric invariants of toric arrangements – in particular, it is not known whether these structures characterize their intersection pattern (one attempt towards closing this gap has been made by considering group actions on semimatroids [14]).