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Hardy space estimates for limited ranges of Muckenhoupt weights

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In this article, we give a full necessary and sufficient set of conditions for a Calderón–Zygmund operator to be bounded on weighted Hardy spaces H^p_w where w is an Muckenhoupt weight and $0 < p < \infty$. In fact, this result is new even when $1 < p < \infty$ since it allows for H^p_w boundedness of an operator when $1 < p < q < \infty$ and $w \in A_q$, where it is possible that $H^p_w \neq L^p_w$. These singular integral results are achieved by proving Littlewood–Paley–Stein square function type estimates from H^p_w into L^p_w for $0 < p < \infty$ and a Muckenhoupt weight w, which are interesting results in their own right. New techniques involving A_∞ weight invariant spaces are also used to prove the weighted estimates for Calderón–Zygmund operators. More precisely, we prove the following BMO type weight invariance properties: for a fixed $s \geq 0$, the weighted Sobolev-BMO spaces $I_s(BMO_w)$ coincide for all $w \in A_\infty$, the weighted $p = \infty$ type Triebel–Lizorkin spaces $\dot{F}^{s,2}_{\infty,w}$ coincide for all $w \in A_\infty$, and these two classes of spaces coincide with each other as well, all of which have comparable norms up to constants depending on an A_p character of the weight $w \in A_\infty$.

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1. Introduction

In this article, we are concerned with boundedness properties of Calderón–Zygmund singular integral operators and Littlewood–Paley–Stein square function operators on weighted Hardy spaces. The primary issue for singular integral operators for us is the continuity of a Calderón–Zygmund operator T from H^p_w into H^p_w for $0 < p < \infty$ and $w \in A_\infty$, for which we give necessary and sufficient conditions; see Theorem 2.10. We also prove new results for square function operators from H^p_w into L^p_w for $w \in A_q$ where $p < q$; see Theorems 2.7 and 2.8. Our approach to these problems uses Muckenhoupt weight invariance properties of BMO and Sobolev-BMO spaces; see Theorem 2.11. In fact, the use of BMO weight invariant properties in this way provides a new way to prove L^p_w type estimates for operators. We will return to this topic at the end of the article for a more in depth discussion.

There is a lot known about Hardy space H^p estimates for Calderón–Zygmund operators, going back to the groundbreaking work of Stein and Weiss [31], Stein [30], and Fefferman and Stein [10], among others. When T is a convolution type operator, things are simplified considerably. It was shown in [10] that if T is a convolution type Calderón–Zygmund operator and is bounded on L^2, then T is also bounded on H^p for $p_0 < p \leq 1$ where $p_0 < 1$ depends on the regularity of the kernel of T. In particular, if the convolution kernel of T is smooth away from the origin, then T is bounded on H^p for all $0 < p \leq 1$. There are also a number of situations where the boundedness properties of a singular integral operator T on weighted Hardy spaces are already known. Still working in the convolution setting, some weighted Hardy space estimates were proved by Lu and Zhu [22]. In that work, the authors prove that if a convolution operator T has a smooth convolution kernel away from the origin and is bounded on L^2, then T is also bounded on H^p_w for all $0 < p < \infty$ and $w \in A_\infty$. One should note here that when $1 < p < \infty$ this result does not collapse to the well-known Lebesgue space theory for singular integral operators. Since the result in [22] allows w to be in any A_q class regardless of the p, it does not follow that $L^p_w = H^p_w$; in particular, when $1 < p < q < \infty$ and $w \in A_q \backslash A_p$ the spaces L^p_w and H^p_w do not coincide. Hence one can conclude from the work in [22] the initially surprising fact that there are convolution operators that are bounded on H^p_w, but not bounded on L^p_w for appropriate selections of $1 < p < \infty$ and $w \in A_\infty$.

In the non-convolution setting, Hardy space estimates are considerably more difficult to prove. Sufficiency results for a non-convolution operator T to be bounded on H^p for $0 < p \leq 1$ were given by Torres [34], Frazier, Torres, and Weiss [14], and Frazier, Han, Jawerth, and Weiss [11]. Full necessary and sufficient theorems for the H^p boundedness of non-convolution Calderón–Zygmund operators were achieved by Alvarez and Milman [1] and the first author and Lu [19]. In [1], the authors give necessary and sufficient conditions for T to be bounded on H^p when p is close to 1 (more precisely when $\frac{n}{n+\gamma} < p \leq 1$ where $0 < \gamma \leq 1$ is the Hölder regularity parameter for the kernel of T), and the full characterization for any $0 < p \leq 1$ was established in [19].