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Bloch functions and asymptotic tail variance ☆

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A B S T R A C T

Let $\mathbf{P}$ denote the Bergman projection on the unit disk $\mathbb{D}$,

$$\mathbf{P}\mu(z) := \int_{\mathbb{D}} \frac{\mu(w)}{(1-z\bar{w})^2}\,\mathrm{d}A(w), \qquad z \in \mathbb{D},$$

where $\mathrm{d}A$ is normalized area measure. We prove that if $|\mu(z)| \le 1$ on $\mathbb{D}$, then the integral

$$I_\mu(a,r) := \int_0^{2\pi} \exp\left\{ a \frac{r^4 |\mathbf{P}\mu(re^{\mathrm{i}\theta})|^2}{\log \frac{1}{1-r^2}} \right\} \frac{\mathrm{d}\theta}{2\pi}, \qquad 0 < r < 1,$$

has the bound $I_\mu(a,r) \le C(a) := 10(1-a)^{-3/2}$ for $0 < a < 1$, irrespective of the choice of the function μ. Moreover, for $a > 1$, no such uniform bound is possible. We interpret the theorem in terms the *asymptotic tail variance* of such a Bergman projection $\mathbf{P}\mu$ (by the way, the asymptotic tail variance induces a seminorm on the Bloch space). This improves upon earlier work of Makarov, which covers the range $0 < a < \frac{\pi^2}{64} = 0.1542\ldots$. We then apply the theorem to obtain an estimate of the universal integral means spectrum for conformal mappings with a k-quasiconformal extension, for $0 < k < 1$. The estimate reads, for $t \in \mathbb{C}$ and $0 < k < 1$,

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$$\mathrm{B}(k,t) \le \begin{cases} \frac{1}{4}k^2|t|^2(1+7k)^2, & \text{for} \quad |t| \le \frac{2}{k(1+7k)^2}, \\ k|t| - \frac{1}{(1+7k)^2}, & \text{for} \quad |t| \ge \frac{2}{k(1+7k)^2}, \end{cases}$$

which should be compared with the conjecture by Prause and Smirnov to the effect that for real t with $|t| \le 2/k$, we should have $\mathrm{B}(k,t) = \frac{1}{4}k^2t^2$.

1. Introduction

1.1. Basic notation

We write $\mathbb{R}$ for the real line, $\mathbb{R}_+ :=]0,+\infty[$ for the positive semi-axis, and $\mathbb{C}$ for the complex plane. Moreover, we write $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ for the extended complex plane (the Riemann sphere). For a complex variable $z = x + \mathrm{i}y \in \mathbb{C}$, let

$$\mathrm{d}s(z) := \frac{|\mathrm{d}z|}{2\pi}, \qquad \mathrm{d}A(z) := \frac{\mathrm{d}x\mathrm{d}y}{\pi},$$

denote the normalized arc length and area measures as indicated. Moreover, we shall write

$$\Delta_z := \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

for the normalized Laplacian, and

$$\partial_z := \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \qquad \bar\partial_z := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right),$$

for the standard complex derivatives; then Δ factors as $\Delta_z = \partial_z\bar\partial_z$. Often we will drop the subscript for these differential operators when it is obvious from the context with respect to which variable they apply. We let $\mathbb{C}$ denote the complex plane, $\mathbb{D}$ the open unit disk, $\mathbb{T} := \partial\mathbb{D}$ the unit circle, and $\mathbb{D}_e$ the exterior disk:

$$\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}, \qquad \mathbb{D}_e := \{z \in \mathbb{C}_\infty : \ |z| > 1\}.$$

More generally, we write

$$\mathbb{D}(z_0, r) := \{z \in \mathbb{C} : \ |z - z_0| < r\}$$

for the open disk of radius r centered at z_0.