Advances in Mathematics 312 (2017) 46–63

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Brill–Noether varieties of k-gonal curves

Nathan Pflueger

Department of Mathematics, Brown University, Box 1917, Providence, RI 02912, United States

ARTICLE INFO

Article history:
Received 30 March 2016
Received in revised form 15 January 2017
Accepted 30 January 2017
Available online xxxx
Communicated by Ravi Vakil

Keywords:
Brill–Noether theory
Special divisors
Gonality
Tropical geometry
Chain of loops

ABSTRACT

We consider a general curve of fixed gonality k and genus g. We propose an estimate $\overline{\rho}_{g,k}(d,r)$ for the dimension of the variety $W^r_d(C)$ of special linear series on C, by solving an analogous problem in tropical geometry. Using work of Coppens and Martens, we prove that this estimate is exactly correct if $k \geq \frac{1}{5}g+2$, and is an upper bound in all other cases. We also completely characterize the cases in which $W^r_d(C)$ has the same dimension as for a general curve of genus g.

1. Introduction

We are concerned in this paper with the varieties $W^r_d(C)$ of special linear series on a general curve of fixed genus g and gonality k over an algebraically closed field K with mild restrictions on the characteristic. See Situation 1.4 for the necessary hypotheses.

For a general curve C of genus g, we have $k = \lfloor \frac{g+3}{2} \rfloor$ and the Brill–Noether theorem [6] says that $\dim W^r_d(C)$ is equal to the Brill–Noether number

E-mail address: pflueger@math.brown.edu.

http://dx.doi.org/10.1016/j.aim.2017.01.027

$$\rho_g(d,r) = g - (r+1)(g-d+r),$$

unless $\rho_g(d,r) < 0$, in which case $W^r_d(C) = \emptyset$.

Our objective is to compute $\dim W^r_d(C)$ for non-generic values of k. We propose a modification $\overline{\rho}_{g,k}(d,r)$ of the Brill–Noether number, incorporating the value of k, and prove the following results. By convention, a negative estimate on $\dim W^r_d(C)$ means that $W^r_d(C)$ is empty.

Theorem 1.1. *In Situation 1.4,* $\dim W^r_d(C) \leq \overline{\rho}_{g,k}(d,r)$.

In characteristic 0, we may combine Theorem 1.1 with a lower bound from results of Coppens and Martens [4], to obtain the following sharp results.

Theorem 1.2. *In Situation 1.4, if* $\operatorname{char} K = 0$ *and either* $k \leq 5$ *or* $k \geq \frac{1}{5}g + 2$, *then* $\dim W^r_d(C) = \overline{\rho}_{g,k}(d,r)$.

Theorem 1.3. *In Situation 1.4, if* $\operatorname{char} K = 0$ *and* $\rho_g(d,r) \geq 0$, *then* $\dim W^r_d(C) = \rho_g(d,r)$ *if and only if* $r = 0$, $g - d + r = 1$, *or* $g - k \leq d - 2r$.

All three theorems use the following notation and hypotheses.

Situation 1.4. Fix nonnegative integers g, k, r, d such that $2 \leq k \leq \frac{g+3}{2}$ and $g-d+r > 0$. Let K be an algebraically closed field, and C be a general k-gonal curve of genus g over K. Also assume that

- if k is odd, then $\operatorname{char} K \neq 2$,
- if $k = 4$ or 10, then $\operatorname{char} K \neq 3$, and
- if $k = 6$, then $\operatorname{char} K \neq 5$.

The hypothesis $g-d+r > 0$ is harmless, since $g-d+r \leq 0$ would imply automatically that $W^r_d(C) = \operatorname{Pic}^d(C)$. The peculiar restrictions on the characteristic of K arise in our proof when we must construct a *tame* morphism of metrized complexes with certain properties. The characteristic 0 assumption in Theorems 1.2 and 1.3 is included because these make use of constructions from [3] and [4], which assume this hypothesis.

1.1. The estimate $\overline{\rho}_{g,k}(d,r)$

The estimate $\overline{\rho}_{g,k}(d,r)$ we refer to above is the following.

Definition 1.5. Let r' denote the minimum of r and $g-d+r-1$. Define

$$\overline{\rho}_{g,k}(d,r) = \max_{\ell \in \{0,1,2,\cdots,r'\}} \left(\rho_g(d, r-\ell) - \ell k\right).$$