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Essential bases and toric degenerations arising from birational sequences

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ABSTRACT

We present a new approach to construct T-equivariant flat toric degenerations of flag varieties and spherical varieties, combining ideas coming from the theory of Newton–Okounkov bodies with ideas originally stemming from PBW-filtrations. For each pair $(S, >)$ consisting of a birational sequence and a monomial order, we attach to the affine variety $G/\!\!/U$ a monoid $\Gamma = \Gamma(S, >)$. As a side effect we get a vector space basis $\mathbb{B}_\Gamma$ of $\mathbb{C}[G/\!\!/U]$, the elements being indexed by Γ. The basis $\mathbb{B}_\Gamma$ has multiplicative properties very similar to those of the dual canonical basis. This makes it possible to transfer the methods of Alexeev and Brion [1] to this more general setting, once one knows that the monoid Γ is finitely generated and saturated.

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1. Introduction

During the recent years, several constructions of T-equivariant flat toric degenerations of flag varieties and spherical varieties have been presented. To name a few examples: the degeneration of SL_n/B by Gonciulea and Lakshmibai in [16]; its interpretation by Kogan and Miller [27] using GIT methods; the approach of Caldero [7] and Alexeev–Brion [1], which uses certain nice properties of the multiplicative behavior of the dual canonical basis, there is the approach of Kaveh [22] and Kiritchenko [25], which has been inspired by the theory of Newton–Okounkov bodies [24,28] (see [2] for constructing toric degenerations using Newton–Okounkov bodies), and there is the approach by Feigin, Fourier and Littelmann in [14], which has been inspired by a conjecture of Vinberg concerning certain filtrations of the enveloping algebra of $\mathcal{U}(\mathfrak{n}^-)$. The starting point for this article was the aim to understand the connection between [25] and [12] (see Example 18).

To be more precise, let us fix some notations. In the following let G be a complex connected reductive algebraic group, we assume that $G \simeq G^{ss} \times (\mathbb{C}^*)^r$, where the semisimple part G^{ss} of G is simply connected. We fix a Cartan decomposition $\operatorname{Lie} G = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{h}$ is a maximal torus and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ is a Borel subalgebra of $\mathfrak{g} = \operatorname{Lie} G$. Correspondingly let U^- and U^+ be the maximal unipotent subgroups of G having $\mathfrak{n}^-$ respectively $\mathfrak{n}^+$ as Lie algebra, and let T be a maximal torus in G with Lie algebra $\mathfrak{h}$ and weight lattice Λ. Denote by Λ^+ the subset of dominant weights. For $\lambda \in \Lambda^+$, let $V(\lambda)$ be the corresponding finite dimensional irreducible representation and v_λ be a highest weight vector. For $\lambda \in \Lambda^+$, let P_λ be the stabilizer of the line $[v_\lambda] \in \mathbb{P}(V(\lambda))$.

The aim of the present article is to exhibit an approach which, in a certain way, unifies the three approaches mentioned above. Let N be the number of positive roots of G. As a first step, we fix a sequence $S = (\beta_1, \ldots, \beta_N)$ of positive roots (they do not have to be pairwise different!). We call the sequence a *birational sequence* for U^- if the product map of the associated unipotent root subgroups:

$$\pi : U_{-\beta_1} \times \cdots \times U_{-\beta_N} \longrightarrow U^-$$

is birational. A simple example for a birational sequence is the case where S is just an enumeration of the set of all positive roots (the PBW-type case), another example is the case where S consists only of simple roots and $w_0 = s_{\beta_1} \cdots s_{\beta_N}$ is a reduced decomposition of the longest word in the Weyl group W of G. For other examples see section 4.2. The second ingredient in our approach is the choice of a weight function $\Psi : \mathbb{N}^N \to \mathbb{N}$ and a refinement of the associated partial order "$>_\Psi$" to a monomial order "$>$" on $\mathbb{N}^N$. We use the birational map π and the total order "$>$" to associate to the variety $G/\!\!/U$ in two different ways a monoid $\Gamma = \Gamma(S, >) \subset \Lambda \times \mathbb{Z}^N$.

Let $(Y, \mathcal{L})$ be a polarized G-variety and $R(Y, \mathcal{L}) = \bigoplus_{n=0}^{\infty} H^0(Y, \mathcal{L}^n)$ be the graded algebra. With the help of the birational map π and the total order "$>$", we introduce a filtration on $R(Y, \mathcal{L})$ (see section 15.2); let $gr\, R(Y, \mathcal{L})$ be the associated graded algebra and $Y_0 = \operatorname{Proj}(gr\, R(Y, \mathcal{L}))$ be the corresponding projective variety. If moreover the polar-

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