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Rationally isomorphic hermitian forms and torsors of some non-reductive groups ☆

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ABSTRACT

Let R be a semilocal Dedekind domain. Under certain assumptions, we show that two (not necessarily unimodular) hermitian forms over an R-algebra with involution, which are *rationally isomorphic* and have isomorphic semisimple *coradicals*, are in fact isomorphic. The same result is also obtained for quadratic forms equipped with an action of a finite group. The results have cohomological restatements that resemble the Grothendieck–Serre conjecture, except the group schemes involved are not reductive. We show that these group schemes are closely related to group schemes arising in Bruhat–Tits theory.

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0. Introduction

Let R be a discrete valuation ring with $2 \in R^{\times}$, and let F be its fraction field. The following theorem is well-known (see for instance [17, Th. 1] for a short proof):

Theorem 0.1. *Let f, f' be two unimodular quadratic forms over R. If f and f' become isomorphic over F, then they are isomorphic over R.*

Over the years, this result has been generalized in many ways; see for instance [12] and [29] for surveys. Many of the generalizations are consequences of the following conjecture:

Conjecture 0.2 *(Grothendieck [18], Serre [39]). Let R be a regular local integral domain with fraction field F. Then for every reductive group scheme $\mathbf{G}$ over R, the induced map*

$$\mathrm{H}^1_{\text{ét}}(R, \mathbf{G}) \to \mathrm{H}^1_{\text{ét}}(F, \mathbf{G})$$

is injective.

The conjecture can also be made for non-connected group schemes whose neutral component is reductive (although it is not true in this generality [12, p. 18]); a widely studied case is the orthogonal group and its forms.

To see the connection to Theorem 0.1, fix a unimodular quadratic space (P, f) and let $\mathbf{U}(f)$ denote the group scheme of isometries of f (the isometries of f are the R-points of $\mathbf{U}(f)$, denoted $U(f)$). Then isomorphism classes of unimodular quadratic forms on the R-module P correspond to $\mathrm{H}^1_{\text{ét}}(R, \mathbf{U}(f))$ (see for instance [22, Ch. III]). Thus, verifying the conjecture for $\mathbf{U}(f)$ implies Theorem 0.1. In this special case, the conjecture was proved when $\dim R \leq 2$ ([26, Cor. 2]) or R contains a field ([27, Th. 9.2]).

The general Grothendieck–Serre conjecture was recently proved by Fedorov and Panin in case R contains a field k; see [16] for the case where k is infinite and [30] for the case where k is finite. Many special cases were known before; see [16] and the references therein. In particular, Nisnevich [25] proved the conjecture when $\dim R = 1$.

Recently, Theorem 0.1 was extended in a different direction by Auel, Parimala and Suresh [1]. Let R denote a semilocal Dedekind domain with $2 \in R^{\times}$ henceforth. A quadratic form f over R has *simple degeneration of multiplicity* 1 if its determinant is square free in R. They show:

Theorem 0.3 *([1, Cor. 3.8]). Let f, f' be two quadratic forms over R having simple degeneration of multiplicity one. If f and f' are isomorphic over F, then they are isomorphic over R.*

Note that the forms f, f' in the theorem may be non-unimodular. When this is the case, they can still be viewed as elements of $\mathrm{H}^1_{\text{ét}}(R, \mathbf{U}(f))$, but $\mathbf{U}(f)$ no longer has a re-