Advances in Mathematics 312 (2017) 185–197

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Finite topology minimal surfaces in homogeneous three-manifolds

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ARTICLE INFO

Article history:
Received 19 October 2016
Accepted 13 March 2017
Available online xxxx
Communicated by Camillo De Lellis

MSC:
primary 53A10
secondary 49Q05, 53C42

Keywords:
Minimal surface
Stability
Minimal lamination
Minimal parking garage structure
Injectivity radius

ABSTRACT

We prove that any complete, embedded minimal surface M with finite topology in a homogeneous three-manifold N has positive injectivity radius. When one relaxes the condition that N be homogeneous to that of being locally homogeneous, then we show that the closure of M has the structure of a minimal lamination of N. As an application of this general result we prove that any complete, embedded minimal surface with finite genus and a countable number of ends is compact when the ambient space is $\mathbb{S}^3$ equipped with a homogeneous metric of nonnegative scalar curvature.

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[1] This material is based upon work for the NSF under Award No. DMS-1309236. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF.

[2] The second author was supported in part by the MINECO/FEDER grant no. MTM2011-22547 and MTM2014-52368-P.

http://dx.doi.org/10.1016/j.aim.2017.03.015

1. Introduction

In this paper we apply the Local Picture Theorem on the Scale of Topology, which is Theorem 1.1 in [7] (see Theorem 1.5 below for the statement of this result in the finite genus setting), to prove that the injectivity radii of certain minimal surfaces in certain Riemannian three-manifolds are never zero.

Theorem 1.1. *Let N be a complete, locally homogeneous*[3] *three-manifold with positive injectivity radius. Then, every complete, embedded minimal surface of finite topology in N has positive injectivity radius.*

In the case that the ambient three-manifold N is isometric to $\mathbb{S}^2 \times \mathbb{R}$ with a scaling of its standard product metric, then this result follows from Theorem 15 in [12] where Meeks and Rosenberg also applied Theorem 1.1 in [7] to prove the stronger property that such minimal surfaces have bounded second fundamental form, linear ambient area growth and so they are also proper in $\mathbb{S}^2 \times \mathbb{R}$. For background material on the geometry and classification of homogeneous three-manifolds, see [6].

The next result removes the positive injectivity radius assumption for the ambient space N. The conclusion that we obtain in this setting is also weaker than the one in Theorem 1.1, as follows from the Minimal Lamination Closure Theorem in [12].

Corollary 1.2. *If M is a complete embedded minimal surface of finite topology in a complete, locally homogeneous three-manifold N, then the closure $\overline{M}$ has the structure of a minimal lamination of N. Furthermore:*

1. *Each limit leaf of $\overline{M}$ is stable (more precisely, the two-sided cover of the leaf is stable).*
2. *If N has positive scalar curvature, then M is proper in N.*
3. *If N is simply connected and has nonnegative scalar curvature, then M is proper in N.*
4. *If N is the round three-sphere $\mathbb{S}^3$, then M is compact.*

Remark 1.3. Item 1 of Corollary 1.2 still holds without the hypothesis on N to be locally homogeneous. On the other hand, it can be shown that there exists a Riemannian metric of positive scalar curvature on the three-sphere that admits a complete embedded minimal plane whose closure does not admit the structure of a minimal lamination (see e.g., [1]). Hence, our hypothesis that N is locally homogeneous is necessary for items 2, 3 of Corollary 1.2 to hold.

[3] A Riemannian manifold N is *locally homogeneous* if given two points in N, balls of the same sufficiently small radius centered at these points are isometric.