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Boutet de Monvel operators on Lie manifolds with boundary

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ABSTRACT

We introduce and study a general pseudodifferential calculus for boundary value problems on a class of non-compact manifolds with boundary (so-called Lie manifolds with boundary). This is accomplished by constructing a suitable generalization of the Boutet de Monvel calculus for boundary value problems. The data consists of a compact manifold with corners M that is endowed with a Lie structure of vector fields $2\mathcal{V}$, a so-called Lie manifold. The manifold M is split into two equal parts X_+ and X_- which intersect in an embedded hypersurface $Y \subset X_{\pm}$. Our goal is to describe a transmission Boutet de Monvel calculus for boundary value problems compatible with the structure of Lie manifolds. Starting with the example of b-vector fields, we show that there are two groupoids integrating the Lie structures on M and on Y, respectively. These two groupoids form a bibundle (or a groupoid correspondence) and, under some mild assumptions, these groupoids are Morita equivalent. With the help of the bibundle structure and canonically defined manifolds with corners, which are blow-ups in particular cases, we define a class of Boutet de Monvel type operators. We then define the representation homomorphism for these operators and show closedness under composition with the help of a representation theorem. Finally, we consider appropriate Fredholm conditions and construct the parametrices for elliptic operators in the calculus.

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1. Introduction

In this work we will enlarge the groupoid pseudodifferential calculus introduced in [29] and develop a general notion of a pseudodifferential calculus for boundary value problems in the framework of Lie groupoids. The most natural approach seems to be along the lines of the Boutet de Monvel calculus. Boutet de Monvel's calculus (e.g. [6]) was established in 1971. This calculus provides a convenient and general framework to study the classical boundary value problems. At the same time parametrices are contained in the calculus and it is closed under composition of elements.

1.1. Overview

In our case, consider the following data: a Lie manifold $(X, \mathcal{V})$ with boundary Y, which is an embedded, transversal hypersurface $Y \subset X$ in the compact manifold with corners X and which is a Lie submanifold of X (cf. [2,1]). The *Lie structure* $\mathcal{V} \subset \Gamma(TX)$ is a Lie algebra of smooth vector fields such that $\mathcal{V}$ is a subset of the Lie algebra $\mathcal{V}_b$ of all vector fields tangent to the boundary strata and a finitely generated projective $C^\infty(X)$-module. From X we define the double $M = 2X$ at the hypersurface Y which is a Lie manifold $(M, 2\mathcal{V})$. The corresponding Lie structure $2\mathcal{V}$ on M is such that $\mathcal{V} = \{V_{|X} : V \in 2\mathcal{V}\}$. Transversality of Y in relation to M means that for each given hyperface $F \subset M$ we have

$$T_x M = T_x F + T_x Y, \ x \in Y \cap F. \tag{1}$$

Introduce the following notation for interior and boundary: by ∂M we mean the union of all hyperfaces of the manifold with corners M,

$$M_0 := M \setminus \partial M, \ Y_0 := Y \cap M_0, \ X_0 := X \cap M_0 \text{ and } \partial Y := Y \cap \partial M.$$

For an open hyperface F in M we denote by $\overline{F}$ the closure in M. Denote by $\partial_{reg} F = \partial_{reg} \overline{F} = \overline{F} \cap Y$ the *regular boundary* of F. The hypersurface Y is endowed with a Lie structure as in [1]:

$$\mathcal{W} = \{V_{|Y} : V \in 2\mathcal{V}, V_{|Y} \text{ tangent to } Y\}. \tag{2}$$

We make the following *assumptions: i)* The hypersurface Y is embedded in M in such a way that the boundary faces of Y are in bijective correspondence with the boundary faces of M. This means the map $\mathcal{F}(M) \ni F \mapsto F \cap Y \in \mathcal{F}(Y)$ should be a bijection, where $\mathcal{F}(M), \mathcal{F}(Y)$ denotes the boundary faces of M and Y respectively. *ii)* Secondly, we assume that for the given Lie structure $2\mathcal{V}$ there is an integrating Lie groupoid $\mathcal{G}$ which is Hausdorff, amenable and has the local triviality property: $\mathcal{G}_F \cong F \times F \times G$ for any open face F of M (where G is an isotropy Lie group). Also $\mathcal{G}_{M_0} \cong M_0 \times M_0$ is the pair groupoid on the interior and $\mathcal{A}_{|M_0} \cong TM_0$ the tangent bundle on the interior.