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The relativistic Boltzmann equation for soft potentials

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ABSTRACT

The paper concerns the Cauchy problem on the relativistic Boltzmann equation for soft potentials in a periodic box. We show that the global-in-time solutions around relativistic Maxwellians exist in the weighted L^∞ perturbation framework and also approach equilibrium states in large time in the weighted L^2 framework at the rate of $\exp(-\lambda t^\beta)$ for some $\lambda > 0$ and $\beta \in (0, 1)$. The proof is based on the nonlinear L^2 energy method and nonlinear L^∞ pointwise estimate with appropriate exponential weights in momentum. The results extend those on the classical Boltzmann equation by Caflisch [2,3] and Strain and Guo [31] to the relativistic version, and also improve the recent result on almost exponential time-decay by Strain [28] to the exponential rate. Moreover, we study the propagation of spatial regularity for the obtained solutions and also the large time behavior in the corresponding regular Sobolev space, provided that the spatial derivatives of initial data are bounded, not necessarily small.

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1. Introduction

The relativistic Boltzmann equation, which is a fundamental model describing the motion of fast moving particles in kinetic theory, takes the form of

$$P \otimes \partial_X F = -\mathcal{C}(F, F). \tag{1.1}$$

Here $\otimes$ represents the Lorentz inner product $(+ - - -)$ of 4-vector. As is customary we write $X = (x_0, x)$ with $x \in \mathbb{T}^3$ and $x_0 = -t$, and $P = (p_0, p)$ with momentum $p \in \mathbb{R}^3$ and energy $p_0 = \sqrt{c^2 + |p|^2}$, where c denotes the speed of light. For convenience of presentation, we rewrite (1.1) supplemented with initial data as

$$\partial_t F + \hat{p} \cdot \nabla_x F = \mathcal{Q}(F, F), \quad F(0, x, p) = F_0(x, p), \tag{1.2}$$

with $\mathcal{Q}(F, F) = \mathcal{C}(F, F)/p_0$, where the unknown $F = F(t, x, p)$ stands for the density distribution function of time $t \geq 0$, space $x \in \mathbb{T}^3$ and momentum $p \in \mathbb{R}^3$. Here the dot represents the standard Euclidean dot product, and the normalized velocity of a particle is denoted as

$$\hat{p} = c\frac{p}{p_0} = \frac{p}{\sqrt{1 + |p|^2/c^2}}.$$

It is known that the constant equilibrium state of (1.1) is the global relativistic Maxwellian, also called the Jütter solution, in the form of

$$J(p) = \frac{\exp\{-cp_0/(k_B T)\}}{4\pi c k_B T K_2(c^2/(k_B T))},$$

where $K_2(z) := \frac{z^2}{2}\int_1^\infty e^{-zt}(t^2 - 1)^{3/2} dt$ is the Bessel function, T is temperature and k_B is the Boltzmann's constant. For notational simplicity we normalize all the physical constants to be one. Then the normalized global relativistic Maxwellian takes the form of

$$J(p) = \frac{e^{-p_0}}{4\pi}, \quad p_0 = \sqrt{1 + |p|^2}. \tag{1.3}$$