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Locally toroidal polytopes of rank 6 and sporadic groups

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ABSTRACT

We augment the list of finite universal locally toroidal regular polytopes of type $\{3,3,4,3,3\}$ due to P. McMullen and E. Schulte, adding as well as removing entries. This disproves a related long-standing conjecture. Our new universal polytope is related to a well-known Y-shaped presentation for the sporadic simple group Fi_{22}, and admits $S_4 \times O_8^+(2){:}S_3$ as the automorphism group. We also discuss further extensions of its quotients in the context of Y-shaped presentations. As well, we note that two known examples of finite universal polytopes of type $\{3,3,4,3,3\}$ are related to Y-shaped presentations of orthogonal groups over $\mathbb{F}_2$. Mixing construction is used in a number of places to describe covers and 2-covers.

1. Introduction and results

Presentations of finite sporadic simple groups as quotients of Coxeter groups with diagram $Y_{\alpha\beta\gamma}$, cf. Fig. 1(a), were discovered 30 years ago [6,7,23] and remain a subject of considerable interest, cf. e.g. [24,8,14,3,1,2]. For instance, the sporadic simple group Fi_{22} is a quotient of Y_{332}, cf. Fig. 1(b) modulo relations (2) below, cf. [6, p. 233]. As explained

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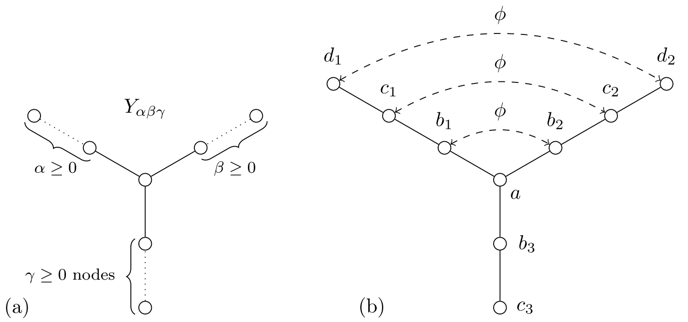

Fig. 1. Y-diagrams; the general case and Y_{332}.

in Section 3 below, these relations afford the automorphism ϕ of the diagram swapping b_1, c_1, and d_1 with, respectively b_2, c_2, and d_2. Thus Fi_{22} has a subgroup isomorphic to a quotient of the Coxeter group with Schläfli symbol $[3, 3, 4, 3, 3]$, i.e. with the string diagram we will also denote by $[3^2, 4, 3^2]$, cf. Fig. 2.

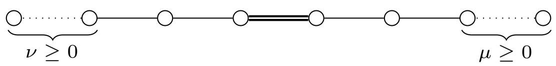

Fig. 2. $[3^{1+\nu}, 4, 3^{1+\mu}]$-diagram.

Groups of this kind arise in the study of *abstract regular polytopes*, for which the book [17] by P. McMullen and E. Schulte is a definite reference. These objects may be viewed as quotients (satisfying *intersection property* (1), cf. [17]) of Coxeter complexes of Coxeter groups with r-node string diagram; here r is called the *rank*. Abstract regular polytopes are classified according to topology of their vertex figures and facets; they are called *locally* X if the latter have topology of X. An up-to-date review on group-theoretic approaches to the problem of classification of abstract regular polytopes may be found e.g. in [5].

Following a problem posed by B. Grünbaum [12, p. 196], particular attention has been paid to finite *locally toroidal* abstract regular polytopes, in this case the rank is bounded from above by 6, cf. [17, Lemma 10A.1].

One of the three rank 6 cases is the case corresponding to the group $[3^2, 4, 3^2]$, cf. [17, Sect. 12D] (or [16, Sect. 7]), where a conjecturally complete list of the finite universal examples is given, see [17, Table 12D1] (which is already in [16, Table V]) and [21, Problem 18]. The main results of the present paper give one more example, missing in that table, and remove erroneous infinite series of examples, of which only first terms actually exist.

In more detail, the facets $\{3, 3, 4, 3\}_{\mathbf{s}}$ and vertex figures $\{3, 4, 3, 3\}_{\mathbf{t}}$ here correspond to nontrivial finite quotients $[3, 3, 4, 3]_{\mathbf{s}}$ and $[3, 4, 3, 3]_{\mathbf{t}}$ of the affine Coxeter group $\tilde{F}_4$, i.e. the groups $[3, 3, 4, 3]$ and $[3, 4, 3, 3]$, with normal Abelian subgroups either of the form q^4 or $q^2 \times (2q)^2$, with $q \geq 2$, where we follow notation from [6] to denote the direct product