Advances in Mathematics 312 (2017) 473–546

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Embeddings of operator ideals into $\mathcal{L}_p$-spaces on finite von Neumann algebras ☆

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ARTICLE INFO

Article history:
Received 16 March 2015
Received in revised form 28 March 2017
Accepted 28 March 2017
Available online xxxx
Communicated by Dan Voiculescu

MSC:
primary 46L53
secondary 81S25

Keywords:
Noncommutative independence
Banach space embeddings

ABSTRACT

Let $\mathcal{L}(H)$ be the $*$-algebra of all bounded operators on an infinite dimensional Hilbert space H and let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ be an ideal in $\mathcal{L}(H)$ equipped with a Banach norm which is distinct from the Schatten–von Neumann ideal $\mathcal{L}_p(\mathcal{H})$, $1 \le p < 2$. We prove that $\mathcal{I}$ isomorphically embeds into an L_p-space $\mathcal{L}_p(\mathcal{R})$, $1 \le p < 2$ (here, $\mathcal{R}$ is the hyperfinite II_1-factor) if its commutative core (that is, Calkin space for $\mathcal{I}$) isomorphically embeds into $L_p(0,1)$. Furthermore, we prove that an Orlicz ideal $\mathcal{L}_M(H) \neq \mathcal{L}_p(H)$ isomorphically embeds into $\mathcal{L}_p(\mathcal{R})$, $1 \le p < 2$, if and only if it is an interpolation space for the Banach couple $(\mathcal{L}_p(H), \mathcal{L}_2(H))$. Finally, we consider isomorphic embeddings of $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ into L_p-spaces associated with arbitrary finite von Neumann algebras.

☆ The first author is partially supported by NSF grant DMS-1201886. The second and third authors are supported by ARC.

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http://dx.doi.org/10.1016/j.aim.2017.03.031
0001-8708/

1. Introduction

Classifying subspaces of Banach spaces is in general a very difficult task, and so far can only be achieved for very special examples of classical function spaces. A beautiful example is Dacuhna-Castelle characterization of subspaces of $L_1(0,1)$ with a symmetric basis, as average of Orlicz spaces. This characterization is genuinely based on the analysis of independent copies and, hence, is probabilistic in nature. Noncommutative function spaces appear naturally in operator theory and noncommutative geometry, and hence it is reasonable to expect a similar striking result for symmetric subspaces of noncommutative L_1-spaces or even noncommutative L_p-spaces in the interesting range for $1 \leq p < 2$. Again M. Kadec [36] discovery of q-stable random variables resulting in embedding l_q into $L_p(0,1)$ serves as inspiration for our results. However, the noncommutative notion of independence is much more ambiguous in the noncommutative setting. Therefore we prefer to use more functional analytic tools, in particular (several versions) of the Kruglov operator introduced in [5] (see [45] for an alternative approach using Poisson processes going back to [31]).

A detailed study of symmetric subspaces of L_1 was done by J. Bretagnolle and D. Dacunha-Castelle (see [15,16,19]). They proved that, for every given mean zero $f \in L_p$, the sequence $\{f_k\}_{k=1}^{\infty}$ of its independent copies is equivalent, in L_p, to the standard basis of some Orlicz sequence space l_M [19, Theorem 1, p. X.8]. Later some of these results were independently rediscovered by M. Braverman [12,14]. Note that the methods used in [15,16,19,12,14] depend heavily on the techniques related to the theory of random processes. In a recent paper [8], a different approach based on methods and ideas from the interpolation theory of operators and the usage of so-called Kruglov operator [4–7] is suggested.

The main topic of this paper is the study of (symmetric) subspaces of the noncommutative analogue of the space $L_p(0,1)$, i.e. the space $\mathcal{L}_p(\mathcal{R},\tau)$ of τ-measurable affiliated with the hyperfinite II$_1$-factor $\mathcal{R}$ of finite p-norm. Indeed, the hyperfinite and finite von Neumann algebra $\mathcal{R}$ is equipped with a unique tracial state τ and the couple $(\mathcal{R},\tau)$ may be considered as a natural noncommutative analogue of the pair $(L_\infty(0,1), dm)$ (here, dm is the Lebesgue measure). In particular, the trace τ is normal and faithful (see next section for details). More generally, closed τ-measurable operators affiliated to $\mathcal{R}$ form a $*$-algebra $\mathcal{S}(\mathcal{R},\tau)$ which is analogous to the algebra $L_0(0,1)$ of all (unbounded) Lebesgue measurable functions on $(0,1)$. The precise definition of the L_p-spaces associated with $\mathcal{R}$ mimics the classical one

$$\mathcal{L}_p(\mathcal{R}) := \{A \in \mathcal{S}(\mathcal{R},\tau) : \ \tau(|A|^p) < \infty\}.$$

The separable space $\mathcal{L}_p(\mathcal{R})$ plays a role in the class of noncommutative L_p-spaces associated with semifinite von Neumann algebras (see e.g. [51]) similar to the role of the space $L_p(0,1)$ in the class of all L_p-spaces on σ-finite measure spaces.