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Orientability for gauge theories on Calabi–Yau manifolds

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We study orientability issues of moduli spaces from gauge theories on Calabi–Yau manifolds. Our results generalize and strengthen those for Donaldson–Thomas theory on Calabi–Yau manifolds of dimensions 3 and 4. We also prove a corresponding result in the relative situation which is relevant to the gluing formula in DT theory.

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1. Introduction

Donaldson invariants count anti-self-dual connections on closed oriented 4-manifolds [17]. The definition requires an orientability result proved by Donaldson in [19]. Indeed, Donaldson theory fits into a 3-dimensional TQFT structure in the sense of Atiyah [2]. In particular, relative Donaldson invariants for $(X, Y = \partial X)$ take values in the instanton Chern–Simons–Floer (co)homology $HF^*_{CS}(Y)$ [20,48]. The Euler characteristic of $HF^*_{CS}(Y)$ is the Casson invariant which counts flat connections on a closed 3-manifold Y.

As was proposed by Donaldson and Thomas [22], we are interested in the *complexification* of the above theory. Namely, we consider holomorphic vector bundles (or general coherent sheaves) over Calabi–Yau manifolds [49]. The complex analogues of (i) Donaldson invariants, (ii) Chern–Simons–Floer (co)homology $HF^*_{CS}(Y)$, and (iii) Casson invariants are (i) DT_4 invariants, (ii) DT_3 (co)homology $H^*_{DT_3}(Y)$, and (iii) DT_3 invariants.

As a complexification of Casson invariants, Thomas defined Donaldson–Thomas invariants for Calabi–Yau 3-folds [45]. DT_3 invariants for ideal sheaves of curves are related to many other interesting subjects including the Gopakumar–Vafa conjecture on BPS numbers in string theory [24,26,32] and the MNOP conjecture [35–37,41] which relates DT_3 invariants and Gromov–Witten invariants. The generalization of DT_3 invariants that count strictly semi-stable sheaves is due to Joyce and Song [31] using Behrend's result [5]. Kontsevich and Soibelman proposed refined as well as motivic DT theory for Calabi–Yau 3-categories [33], which was later studied by Behrend, Bryan and Szendröi [6] for Hilbert schemes of points. The wall-crossing formula [33,31] is an important structure for Bridgeland's stability condition [11] and Pandharipande–Thomas invariants [42, 46].

As a complexification of Chern–Simons–Floer theory, Brav, Bussi, Dupont, Joyce and Szendroi [9], Kiem and Li [32] recently defined a cohomology theory on Calabi–Yau 3-folds whose Euler characteristic is the DT_3 invariant. The point is that moduli spaces of simple sheaves on Calabi–Yau 3-folds are locally critical points of holomorphic functions [10,31], and we could consider perverse sheaves of vanishing cycles of these functions. They glued these local perverse sheaves and defined DT_3 cohomology as its hypercohomology. In general, gluing these perverse sheaves requires a choice of a square root of the determinant line bundle of the moduli space. Nekrasov and Okounkov proved its existence in [40]. The square root is called an orientation data if it is furthermore compatible with wall-crossing (or Hall algebra structure) [33] whose existence was proved by Hua on simply-connected torsion-free CY_3 [27].

As a complexification of Donaldson theory, Borisov and Joyce [7] and the authors [13,14] developed DT_4 invariants (or 'holomorphic Donaldson invariants') which count stable sheaves on Calabi–Yau 4-folds. To define the invariants, we need an orientability result, which was obtained by the authors in [14] for Calabi–Yau 4-fold X which satisfies