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Intrinsic random walks and sub-Laplacians in sub-Riemannian geometry

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A B S T R A C T

On a sub-Riemannian manifold we define two types of Laplacians. The *macroscopic Laplacian* Δ_ω, as the divergence of the horizontal gradient, once a volume ω is fixed, and the *microscopic Laplacian*, as the operator associated with a sequence of geodesic random walks. We consider a general class of random walks, where *all* sub-Riemannian geodesics are taken in account. This operator depends only on the choice of a complement $\mathbf{c}$ to the sub-Riemannian distribution, and is denoted by $L^{\mathbf{c}}$.
We address the problem of equivalence of the two operators. This problem is interesting since, on equiregular sub-Riemannian manifolds, there is always an intrinsic volume (e.g. Popp's one $\mathcal{P}$) but not a canonical choice of complement. The result depends heavily on the type of structure under investigation:

- On contact structures, for every volume ω, there exists a unique complement $\mathbf{c}$ such that $\Delta_\omega = L^{\mathbf{c}}$.
- On Carnot groups, if H is the Haar volume, then there always exists a complement $\mathbf{c}$ such that $\Delta_H = L^{\mathbf{c}}$. However this complement is *not unique* in general.
- For quasi-contact structures, in general, $\Delta_{\mathcal{P}} \neq L^{\mathbf{c}}$ for any choice of $\mathbf{c}$. In particular, $L^{\mathbf{c}}$ is not symmetric with respect to Popp's measure. This is surprising especially in dimension 4 where, in a suitable sense, $\Delta_{\mathcal{P}}$ is the unique intrinsic macroscopic Laplacian.

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A crucial notion that we introduce here is the *N-intrinsic volume*, i.e. a volume that depends only on the set of parameters of the nilpotent approximation. When the nilpotent approximation does not depend on the point, a N-intrinsic volume is unique up to a scaling by a constant and the corresponding N-intrinsic sub-Laplacian is unique. This is what happens for dimension less than or equal to 4, and in particular in the 4-dimensional quasi-contact structure mentioned above.
Finally, we prove a general theorem on the convergence of families of random walks to a diffusion, that gives, in particular, the convergence of the random walks mentioned above to the diffusion generated by $L^{\mathbf{c}}$.

Contents