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Tensor triangular geometry for classical Lie superalgebras

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ABSTRACT

Tensor triangular geometry as introduced by Balmer [3] is a powerful idea which can be used to extract the ambient geometry from a given tensor triangulated category. In this paper we provide a general setting for a compactly generated tensor triangulated category which enables one to classify thick tensor ideals and the Balmer spectrum. For the general linear Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ we construct a Zariski space from a detecting subalgebra of $\mathfrak{g}$ and demonstrate that this topological space governs the tensor triangular geometry for the category of finite dimensional $\mathfrak{g}$-modules which are semisimple over $\mathfrak{g}_{\bar{0}}$.

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1. Introduction

1.1. A predominant theme in representation theory is the utilization of ambient geometric structures to study a given module category. For a symmetric monoidal tensor triangulated category, $\mathbf{K}$, Balmer [3] first introduced the idea of tensor triangular geometry and used it to show that the underlying geometry can be revealed through the use of the tensor structure. He defined the notion of a prime ideal and constructed the (Balmer) spectrum, $\mathrm{Spc}(\mathbf{K})$, thus allowing one to study these categories from the viewpoint of commutative algebra and algebraic geometry. In particular he showed that a scheme can be reconstructed from an associated tensor triangulated category via his construction. Another important example occurs when $\mathbf{K} = \mathrm{Stab}(\mathrm{mod}(G))$ is the stable module category of finitely generated modules for a finite group scheme G over a field k. In this setting there exists a homeomorphism between $\mathrm{Spc}(\mathbf{K})$ and $\mathrm{Proj}(R) := \mathrm{Proj}\,\mathrm{Spec}((R))$ where $R = \mathrm{H}^{2\bullet}(G,k)$ is the cohomology ring for G. There are typically many support variety theories (support data) for $\mathbf{K}$, with $\mathrm{Spc}(\mathbf{K})$ being the universal (or "final") support variety theory. Determining $\mathrm{Spc}(\mathbf{K})$ is closely connected to the classification of thick tensor ideals and recovers geometry hidden within $\mathbf{K}$. Computing the spectrum of interesting tensor triangulated categories remains one of the most fundamental questions in the subject. See [4] for further discussion of the spectrum.

Methods for classifying thick tensor ideals originated in the work of Hopkins [26] in the context of the derived category of bounded complexes of finitely generated projective modules over a commutative Noetherian ring. Benson, Carlson, and Rickard [6] later studied this question for the stable module category of a group algebra. In their work it became apparent that infinitely generated modules would play a key role in the classification. Specifically, they provide a systematic treatment using ideas from homotopy theory and Rickard's idempotent modules to demonstrate that the thick tensor ideals for the stable module category of a finite group are in correspondence with the specialization closed sets in $\mathrm{Proj}(R)$. The description of $\mathrm{Spc}(\mathbf{K})$ can then be deduced from this fact. In [23] Friedlander and Pevtsova introduced a new approach using so-called π-points and extended the above description of $\mathrm{Spc}(\mathbf{K})$ to arbitrary finite group schemes. A fundamental new idea recently introduced by Benson, Iyengar and Krause [8,9] is that of having a commutative ring R stratify the category $\mathbf{K}$. This allowed them to further develop the theory while recovering the aforementioned results in the case of finite groups.

1.2. The authors of this paper initiated a study of classical Lie superalgebras via cohomology and support varieties in [12,11,13,14]. Given a classical Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ over $\mathbb{C}$ let $G_{\bar{0}}$ be the connected reductive algebraic group with $\mathrm{Lie}\,(G_{\bar{0}}) = \mathfrak{g}_{\bar{0}}$. It is natural to consider the category $\mathcal{F} = \mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ of finite dimensional $\mathfrak{g}$-modules which admit a compatible action by $G_{\bar{0}}$ and are completely reducible as $G_{\bar{0}}$-modules. The category $\mathcal{F}$ enjoys many of the features found for finite group schemes except the blocks in $\mathcal{F}$ can have infinitely many irreducible representations. One can use the fact that $\mathcal{F}$