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The porous medium equation on Riemannian manifolds with negative curvature. The large-time behaviour

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A B S T R A C T

We consider nonnegative solutions of the porous medium equation (PME) on Cartan–Hadamard manifolds whose negative curvature can be unbounded. We take compactly supported initial data because we are also interested in free boundaries. We classify the geometrical cases we study into quasi-hyperbolic, quasi-Euclidean and critical cases, depending on the growth rate of the curvature at infinity. We prove sharp upper and lower bounds on the long-time behaviour of the solutions in terms of corresponding bounds on the curvature. In particular we estimate the location of the free boundary. A global Harnack principle follows.
We also present a change of variables that allows to transform radially symmetric solutions of the PME on model manifolds into radially symmetric solutions of a corresponding weighted

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PME on Euclidean space and back. This equivalence turns out to be an important tool of the theory.

1. Introduction and outline of results

This paper is concerned with the porous medium equation (PME for short)

$$\begin{cases} u_t = \Delta u^m & \text{in } M \times \mathbb{R}^+ , \\ u(\cdot, 0) = u_0 & \text{in } M , \end{cases} \tag{1.1}$$

where $m > 1$, Δ denotes the Laplace–Beltrami operator on a Riemannian manifold (M, g) without boundary and the initial datum u_0 is assumed to be nonnegative, bounded and compactly supported. The main assumption on M is that it is a *Cartan–Hadamard manifold*, namely that it is complete, simply connected and has everywhere *nonpositive* sectional curvature.

The study of the PME on such kind of manifolds is quite recent, and the first results in this connection concern the special case in which $M = \mathbb{H}^n$, the n-dimensional hyperbolic space, a manifold having special significance since its sectional curvature is -1 everywhere. In fact, two of the present authors considered recently in [13] the case of the *fast diffusion equation* (namely, (1.1) with $m < 1$) in $\mathbb{H}^n$, proving precise asymptotics for positive solutions. In [24], the last of the present authors constructed and studied the fundamental tool for studying the asymptotic behaviour of general solutions of (1.1) on $\mathbb{H}^n$, namely the *Barenblatt solutions*, which are solutions of (1.1) corresponding to a Dirac delta as initial datum. Two of the most important results of [24] can be summarised as follows:

• The decay estimate

$$\|u(t)\|_\infty \le C \left(\log t/t\right)^{\frac{1}{m-1}} \tag{1.2}$$

holds for all t sufficiently large. This bound is quite different from the corresponding Euclidean one, where the sharp upper bound takes the form $u(x,t) \le Ct^{-\alpha(n)}$, with exponent $\alpha(n) = (m - 1 + (2/n))^{-1}$ which is strictly less than $1/(m-1)$ for all n; the difference is smaller as $n \to \infty$. Estimate (1.2) bears closer similarity to the estimate that one obtains when the problem is posed in a *bounded Euclidean domain* and zero Dirichlet boundary data are assumed, since then the sharp estimate is $u(x,t) \sim C(x)t^{-1/(m-1)}$, differing from the (1.2) only in the logarithmic time correction.

• Solutions corresponding to a compactly supported initial datum are compactly supported for all times, and in particular the free boundary $\mathsf{R}(t)$ of the Barenblatt solution behaves for very large times like $\mathsf{R}(t) \sim \gamma \log t + b$ for some precise constants γ, b. In fact,