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Configuration spaces and polyhedral products ☆

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ABSTRACT

This paper aims to find the most general combinatorial conditions under which a moment–angle complex $(D^2, S^1)^K$ is a co-H-space, thus splitting unstably in terms of its full subcomplexes. In this way we study to which extent the conjecture holds that a moment–angle complex over a Golod simplicial complex is a co-H-space. Our main tool is a certain generalisation of the theory of labelled configuration spaces.

1. Introduction

Polyhedral products have been the subject of quite a bit of interest recently, beginning with their appearance as homotopy theoretical generalisations of various objects stud-

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ied in toric topology. Of particular importance are the polyhedral products $(D^2, S^1)^K$ and $(\mathbb{C}P^\infty, *)^K$, known as *moment–angle complexes* and *Davis–Januszkiewicz spaces* respectively. The homotopy theory of these spaces has many applications – from complex and symplectic geometry (cf. [8,27,26]), to combinatorial and homological algebra (cf. [32,3]). For example, moment–angle manifolds appear as intersection of quadrics or as quasitoric manifolds after taking a certain orbit space, *Stanley–Reisner rings* of simplicial complexes are realised by the cohomology of Davis–Januszkiewicz spaces (equivalently, the equivariant cohomology ring of moment–angle complexes), while the cohomology ring of moment–angle complexes is closely related to the study of the cohomology of local rings (cf. [10,9]). One would like to know how the combinatorics of the underlying simplicial complex K encodes geometrical and topological properties of polyhedral products, and vice-versa. The case of Golod complexes is especially relevant. A ring $R = \mathbf{k}[v_1, \ldots, v_n]/I$ for I a homogeneous ideal is said to be *Golod* if all products and higher Massey products in $\mathrm{Tor}^+_{\mathbf{k}[v_1,\ldots,v_n]}(R, \mathbf{k})$ vanish. Golod [17] showed that the Poincaré series of the homology ring $\mathrm{Tor}_R(\mathbf{k}, \mathbf{k})$ of R is a rational function whenever R is Golod. From the context of combinatorics, a simplicial complex K on vertex set $[n] = \{1, \ldots, n\}$ is said to be *Golod* over $\mathbf{k}$ if the *Stanley–Reisner ring* $\mathbf{k}[K]$ is Golod, and if this is true for all fields $\mathbf{k}$ and $\mathbf{k} = \mathbb{Z}$, we simply say that K is Golod. Fixing $\mathbf{k}$ to be a field or $\mathbb{Z}$, by [9,15,5,22] there are isomorphisms of graded commutative algebras

$$H^*((D^2, S^1)^K; \mathbf{k}) \cong \mathrm{Tor}_{\mathbf{k}[v_1,\ldots,v_n]}(\mathbf{k}[K], \mathbf{k}) \cong \bigoplus_{I \subseteq [n]} \tilde{H}^*(\Sigma^{|I|+1}|K_I|; \mathbf{k})$$

where $\mathbf{k}[K]$ is the Stanley–Reisner ring of K, K_I is the restriction of K to vertex set $I \subseteq [n]$, and the multiplication in the rightmost algebra is realised by maps

$$\iota_{I,J} : |K_{I \cup J}| \longrightarrow |K_I * K_J| \cong |K_I| * |K_J| \simeq \Sigma|K_I| \wedge |K_J|,$$

induced by the canonical inclusions $K_{I\cup J} \longrightarrow K_I * K_J$ whenever I and J are non-empty and disjoint. The Golod condition can then be reinterpreted as $\iota_{I,J}$ inducing trivial maps on $\mathbf{k}$-cohomology for disjoint non-empty I and J together with Massey products vanishing in $H^+((D^2, S^1)^K; \mathbf{k})$. This topological interpretation of the Golod condition has been a starting point for applying the homotopy theory of moment–angle complexes to the problem of determining which simplicial complexes K are Golod, see [19] for example. In the opposite direction, the cohomology of a moment–angle complex $(D^2, S^1)^K$ takes its simplest algebraic form when we restrict to Golod K. Golod complexes are therefore a natural starting point for studying the homotopy types of moment–angle complexes.

Considerable work has been done on the homotopy theory of moment–angle complexes over Golod complexes [18–20,24,23,21], culminating in the following conjectured topological characterisation of the Golod complexes.