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Non-compact Newton boundary and Whitney equisingularity for non-isolated singularities

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ABSTRACT

In an unpublished lecture note, J. Briançon observed that if $\{f_t\}$ is a family of isolated complex hypersurface singularities such that the Newton boundary of f_t is independent of t and f_t is non-degenerate, then the corresponding family of hypersurfaces $\{f_t^{-1}(0)\}$ is Whitney equisingular (and hence topologically equisingular). A first generalization of this assertion to families with non-isolated singularities was given by the second author under a rather technical condition. In the present paper, we give a new generalization under a simpler condition.

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1. Introduction

Let $(t, \mathbf{z}) := (t, z_1, \ldots, z_n)$ be coordinates for $\mathbb{C} \times \mathbb{C}^n$, let U be an open neighbourhood of $\mathbf{0} \in \mathbb{C}^n$ and D be an open disc centered at $0 \in \mathbb{C}$; finally, let

$$f\colon (D \times U, D \times \{\mathbf{0}\}) \to (\mathbb{C}, 0),\ \ (t, \mathbf{z}) \mapsto f(t, \mathbf{z}),$$

be a polynomial function. As usual, we write $f_t(\mathbf{z}) := f(t, \mathbf{z})$ and we denote by $V(f_t)$ the hypersurface in $U \subseteq \mathbb{C}^n$ defined by f_t. We are interested in the local structure of the singular loci of the hypersurfaces $V(f_t)$ at the origin $\mathbf{0} \in \mathbb{C}^n$ as the parameter t varies from a "small" non-zero value $t_0 \neq 0$ to $t = 0$. More precisely, we are looking for easy-to-check conditions on the members f_t of the family $\{f_t\}$ that guarantee equisingularity (in a sense to be specified) for the corresponding family of hypersurfaces $\{V(f_t)\}$.

In an unpublished lecture note [1], J. Briançon made the following observation.

Assertion 1.1 *(Briançon). Suppose that for all t sufficiently small, the following three conditions are satisfied:*

(1) *f_t has an isolated singularity at the origin $\mathbf{0} \in \mathbb{C}^n$;*
(2) *the Newton boundary $\Gamma(f_t; \mathbf{z})$ of f_t at $\mathbf{0}$ with respect to the coordinates $\mathbf{z}$ is independent of t;*
(3) *f_t is non-degenerate (in the sense of the Newton boundary as in [3,6]).*

Then the family of hypersurfaces $\{V(f_t)\}$ is Whitney equisingular.

We say that a family $\{V(f_t)\}$ of (possibly non-isolated) hypersurface singularities is *Whitney equisingular* if there exists a Whitney stratification of the hypersurface $V(f) := f^{-1}(0)$ in an open neighbourhood $\mathscr{U}$ of the origin $(0, \mathbf{0}) \in \mathbb{C} \times \mathbb{C}^n$ such that the t-axis $\mathscr{U} \cap (D \times \{\mathbf{0}\})$ is a stratum. By "Whitney stratification" we mean a Whitney stratification in the sense of [2]—that is, we do not require that the frontier condition holds. However, note that if $\mathscr{S}$ is Whitney stratification of $\mathscr{U} \cap V(f)$ with the t-axis as a stratum, then so is the partition $\mathscr{S}^c$ consisting of the connected components of the strata of $\mathscr{S}$; moreover, $\mathscr{S}^c$ satisfies the frontier condition (see [2] for details). Whitney equisingularity is quite a strong form of equisingularity. Combined with the Thom–Mather first isotopy theorem (cf. [2,4,12]), it implies topological equisingularity. Here, we say that the family $\{V(f_t)\}$ is *topologically equisingular* if for all sufficiently small t, there is an open neighbourhood $U_t \subseteq U$ of $\mathbf{0} \in \mathbb{C}^n$ together with a homeomorphism $\varphi_t\colon (U_t, \mathbf{0}) \to (\varphi_t(U_t), \mathbf{0})$ such that $\varphi_t(V(f_0) \cap U_t) = V(f_t) \cap \varphi_t(U_t)$. Note that a family of *isolated* hypersurface singularities (as in Assertion 1.1) is Whitney equisingular if and only if $V(f) \setminus (D \times \{\mathbf{0}\})$ is smooth and Whitney (b)-regular over $D \times \{\mathbf{0}\}$ in an open neighbourhood of the origin in $\mathbb{C} \times \mathbb{C}^n$. Here, it is worth to observe that, in general, even if the smooth part of $V(f)$ is Whitney (b)-regular along the t-axis, the family of