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Endpoint estimates for one-dimensional oscillator integral operators

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ABSTRACT

The one-dimensional oscillatory integral operator associated to a real analytic phase S is given by

$$T_\lambda f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} \chi(x,y) f(y) dy.$$

In their fundamental work, Phong and Stein established sharp L^2 estimates for T_λ. The goal of this paper is to extend their results to all endpoints. In particular, we obtain a complete characterization for the mapping properties for T_λ on $L^p(\mathbb{R})$. More precisely, we show that $\|T_\lambda f\|_p \lesssim |\lambda|^{-\alpha} \|f\|_p$ holds for some $\alpha > 0$ if and only if $(\frac{1}{\alpha p}, \frac{1}{\alpha p'})$ lies in the reduced Newton polygon of S.

1. Introduction

The very well-known and extremely useful tool in one-dimensional analysis is the following van der Corput lemma (see [1,33]):

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Lemma 1.1. *For any real-valued function $u \in C^k(I)$ on some closed interval $I \subset \mathbb{R}$, if $|u^{(k)}(x)| \neq 0$ on I (assuming u' is monotone when $k = 1$),*

$$\left| \int_I e^{i\lambda u(x)} dx \right| \leq C_k |\lambda|^{-\frac{1}{k}}, \quad \forall \lambda \ \in \mathbb{R}.$$

This estimate shares the following two remarkable features: sharpness (the decay is best possible) and uniformity/stability (the constant C_k depends only on the lower bound of $|u^{(k)}|$ in I but not on other assumptions concerning u). In several variables (we mainly focus on two variables here) one has the following (see [33]): suppose $\chi \in C_0^\infty(\mathbb{R}^2)$ and S is a real-valued function so that for some $(k, l) \in \mathbb{N}^2$ not equal to $(0, 0)$,

$$\left| \frac{\partial^{k+l} S}{\partial_x^k \partial_y^l}(x, y) \right| \neq 0 \quad \text{for all} \quad (x, y) \in \text{supp } \chi. \tag{1.1}$$

Then

$$\left| \iint_{\mathbb{R}^2} e^{i\lambda S(x,y)} \chi(x, y) \, dxdy \right| \leq C(S) \cdot |\lambda|^{-\frac{1}{k+l}}. \tag{1.2}$$

However, this two dimensional analogue is less satisfying for the estimate is neither sharp in general (consider $S(x, y) = x^k y^l$, for example) nor uniform ($C(S)$ depends on higher derivatives of the phase). There has been significant interest in the harmonic analysis literature to develop a robust and general theory of high-dimensional oscillatory integrals that shares the above two features. However, progress on this problem has been slow, because, among many other reasons, the singularities of the phases involved may themselves be substantially more complicated. Much progress has been made in the feature of uniformity/stability. Results in this category include [1–3,8,10,11,17,25,26]. The goal of the present paper is trying to understand the other feature, namely the feature of sharpness. More precisely, we are interested in an oscillatory integral model that is intrinsically associated to (1.1), in the sense its sharp decay rate estimates are essentially equivalent to the assumption (1.1), but restrict ourself to analytic phases.

Throughout the rest of this paper, χ denotes a function belonging to $C_0^\infty(\mathbb{R}^2)$ and the phase S is real analytic in supp χ, in the sense S is locally equal to its Taylor expansion in supp χ. The subjects under consideration are the $1 + 1$ dimensions oscillatory integral operator

$$T_\lambda f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} \chi(x, y) f(y) dy. \tag{1.3}$$

We are interested in decay rates (as $\lambda \to \pm\infty$) for the norm of T_λ as an operator that maps $L^p(\mathbb{R})$ into itself for all $p \geq 1$. For convenience, we use $\|T_\lambda\|_p$ to denote this norm