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Double canonical bases ☆

Arkady Berenstein [a], Jacob Greenstein [b,*]

[a] *Department of Mathematics, University of Oregon, Eugene, OR 97403, USA*
[b] *Department of Mathematics, University of California, Riverside, CA 92521, USA*

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ABSTRACT

We introduce a new class of bases for quantized universal enveloping algebras $U_q(\mathfrak{g})$ and other doubles attached to semisimple and Kac–Moody Lie algebras. These bases contain dual canonical bases of upper and lower halves of $U_q(\mathfrak{g})$ and are invariant under many symmetries including all Lusztig's symmetries if $\mathfrak{g}$ is semisimple. It also turns out that a part of a double canonical basis of $U_q(\mathfrak{g})$ spans its center and consists of higher Casimirs which suggests physical applications.

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* Corresponding author.
E-mail addresses: arkadiy@math.uoregon.edu (A. Berenstein), jacob.greenstein@ucr.edu (J. Greenstein).

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1. Introduction and main results

The goal of this paper is to construct a canonical basis $\mathbf{B}_{\mathfrak{g}}$ of a quantized enveloping algebra $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is a semisimple or a Kac–Moody Lie algebra. For instance, if $\mathfrak{g} = \mathfrak{sl}_2$, then $\mathbf{B}_{\mathfrak{g}}$ is given by

$$\mathbf{B}_{\mathfrak{sl}_2} = \{q^{n(m_- - m_+)} K^n C^{(m_0)} F^{m_-} E^{m_+} \,|\, n \in \mathbb{Z},\, m_0, m_\pm \in \mathbb{Z}_{\geq 0},\, m_- \cdot m_+ = 0\}, \quad (1.1)$$

where we used a slightly non-standard presentation of $U_q(\mathfrak{sl}_2)$ obtained from the more familiar one by rescaling generators $E \mapsto (q^{-1} - q)E$, $F \mapsto (q - q^{-1})F$:

$$U_q(\mathfrak{sl}_2) := \langle E, F, K^{\pm 1} \,:\, KEK^{-1} = q^2 E,\, KFK^{-1} = q^{-2} F,$$
$$EF - FE = (q^{-1} - q)(K - K^{-1})\rangle.$$

Here the $C^{(m)}$ are central elements of $U_q(\mathfrak{sl}_2)$ defined by $C^{(0)} = 1$, $C = C^{(1)} = EF - q^{-1}K - qK^{-1} = FE - qK - q^{-1}K^{-1}$ and $C \cdot C^{(m)} = C^{(m+1)} + C^{(m-1)}$ for $m \geq 1$.

We call $\mathbf{B}_{\mathfrak{sl}_2}$ double canonical because of the following remarkable properties (we will explain in §4.1 the reason why we must use Chebyshev polynomials $C^{(m)}$ instead of C^m).