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The resolution of the Yang–Mills Plateau problem in super-critical dimensions

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ABSTRACT

We study the minimization problem for the Yang–Mills energy under fixed boundary connection in supercritical dimension $n \geq 5$. We define the natural function space $\mathcal{A}_G$ in which to formulate this problem in analogy to the space of integral currents used for the classical Plateau problem. The space $\mathcal{A}_G$ can be also interpreted as a space of weak connections on a "real measure theoretic version" of reflexive sheaves from complex geometry.
We prove the existence of weak solutions to the Yang–Mills Plateau problem in the space $\mathcal{A}_G$.
We then prove the optimal regularity result for solutions of this Plateau problem. On the way to prove this result we establish a Coulomb gauge extraction theorem for weak curvatures with small Yang–Mills density. This generalizes to the general framework of weak L^2 curvatures previous works of Meyer–Rivière and Tao–Tian in which respectively a strong approximability property and an admissibility property were assumed in addition.

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1. Introduction

1.1. A nonintegrable Plateau problem

Consider a smooth compact Riemannian n-manifold M with boundary and let G be a compact connected simply connected non-abelian Lie group with Lie algebra $\mathfrak{g}$. We assume that a principal G-bundle $P \to \partial M$ is fixed over the boundary of M. On P we consider a G-invariant connection ω, which corresponds to an equivariant horizontal n-plane distribution Q (see [26] for notations and definitions).

Analogously to the Plateau problem, we may then ask which is the "most integrable" extension of P, Q to a horizontal distribution on a principal G-bundle over M. By Frobenius' theorem, the condition for integrability in this case is that for any two horizontal G-invariant vector fields X, Y, their lie bracket $[X, Y]$ be again horizontal. The L^2-error to integrability of an extension of Q over M can be measured by taking vertical projections $\mathcal{V}$ of $[X_i, X_j]$ for X_i, X_j varying in an orthonormal basis of Q:

$$\int_M \sum_{i,j} |\mathcal{V}([X_i, X_j])|^2 \ . \tag{1.1}$$

Note that $F(X, Y) = \mathcal{V}([X, Y])$ is known to be a tensor, and F is nothing but the curvature of the connection.

From now on we will work on the vector bundle $E \to M$ associated to the principal bundle G corresponding to a representation of G. The covariant derivative ∇ on E is identified, in a trivialization, and via the implicit action of the representation, with the local expression

$$\nabla \overset{loc}{=} d + A \ ,$$

where A is a $\mathfrak{g}$-valued 1-form on a given chart of M. The structure equation relating curvature to connection takes the form

$$F \overset{loc}{=} dA + A \wedge A \tag{1.2}$$

in a trivialization. Here $\wedge$ represents a tensorization of the usual exterior product of forms with the Lie bracket on $\mathfrak{g}$. In this setting the L^2-error in integrability (1.1) is identified with the *Yang–Mills energy*, which we consider as being a functional of the connection ∇:

$$\mathcal{YM}(\nabla) := \int_M |F_\nabla|^2 \ . \tag{1.3}$$

We observe that, similarly to the area functional in the Plateau problem, $\mathcal{YM}$ has a *large invariance group* given by changing coordinates in the fibers via G. The gauge group can,