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A proof of the $C^{p'}$-regularity conjecture in the plane

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A B S T R A C T

We establish a new oscillation estimate for solutions of nonlinear partial differential equations of elliptic, degenerate type. This new tool yields a precise control on the growth rate of solutions near their set of critical points, where ellipticity degenerates. As a consequence, we are able to prove the planar counterpart of the longstanding conjecture that solutions of the degenerate p-Poisson equation with a bounded source are locally of class $C^{p'} = C^{1,\frac{1}{p-1}}$; this regularity is optimal.

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1. Introduction

In this paper we investigate sharp $C^{1,\alpha}$-regularity estimates for solutions of the degenerate elliptic equation, with a bounded source,

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$$-\Delta_p u = f(x) \in L^\infty(B_1), \qquad p > 2. \tag{1.1}$$

Establishing optimal regularity estimates is quite often a delicate matter and, in particular, $f(x) \in L^\infty$ is known to be a borderline condition for regularity. In the linear, uniformly elliptic case $p = 2$, solutions of

$$-\Delta u = f(x) \in L^\infty(B_1)$$

are locally in $C^{1,\alpha}$, for every $\alpha \in (0,1)$, but may fail to be in $C^{1,1}$. Obtaining such an estimate in specific situations, like free boundary problems, often involves a deep and fine analysis.

In the degenerate setting $p > 2$, the smoothing effects of the operator are far less efficient. Nonetheless, it is well established, see for instance [8,23], that a weak solution to (1.1) is locally of class $C^{1,\beta}$, for some exponent $\beta > 0$ depending on dimension and p. If p' denotes the conjugate of p, i.e.,

$$p + p' = pp',$$

the radial symmetric example

$$-\Delta_p \left(c_p |x|^{p'} \right) = 1$$

sets the limits to the optimal regularity and gives rise to the following well known open problem among experts in the field.

Conjecture *($C^{p'}$-regularity conjecture). Solutions to (1.1) are locally of class $C^{1,\frac{1}{p-1}} = C^{p'}$.*

This problem touches very subtle issues in regularity theory. As mentioned above, the conjecture is not true in the linear, uniformly elliptic setting, $p = 2$, where merely $C^{1,\mathrm{LogLip}}$-estimates are possible. Notice further that a positive answer implies that $|x|^{p'}$ – a function whose p-laplacian is constant (real analytic) – is the least regular among all functions whose p-laplacian is bounded. This is, at first sight, counterintuitive.

We show in this paper that the conjecture holds true provided p-harmonic functions, which are the solutions of the homogeneous counterpart of (1.1), are locally uniformly of class $C^{1,\alpha}$, with

$$\alpha > \frac{1}{p-1}.$$

While this is still open in higher dimensions, it holds true in the plane, thus yielding a full proof of the conjecture in 2-d. The crucial estimate follows from results by Baernstein II and Kovalev in [5], exploiting the fact that the complex gradient of a p-harmonic function in the plane is a K-quasiregular gradient mapping. In a somewhat related issue, let us