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Universality limits for generalized Jacobi measures

Tivadar Danka [1]

Bolyai Institute, University of Szeged, Aradi V. tere 1, H6720 Szeged, Hungary

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A B S T R A C T

In this paper universality limits are studied in connection with measures which exhibit power-type singular behavior somewhere in their support. We extend the results of Lubinsky for Jacobi measures supported on $[-1, 1]$ to generalized Jacobi measures supported on a compact subset of the real line, where the singularity can be located in the interior or at an endpoint of the support. The analysis is based upon the Riemann–Hilbert method, Christoffel functions, the polynomial inverse image method of Totik and the normal family approach of Lubinsky.

1. Introduction

Universality limits for random matrices is an intensively studied topic of mathematics and mathematical physics, which has several applications even outside physics and

E-mail address: tdanka@math.u-szeged.hu.
URL: http://www.tivadardanka.com.

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mathematics. For ensembles of $n \times n$ Hermitian random matrices invariant under unitary conjugation, a connection with orthogonal polynomials can be established. If the eigenvalue distribution is given by

$$p(x_1, \dots, x_n) = \frac{1}{Z_n} \prod_{1 \le i < j \le n} |x_j - x_i|^2 \prod_{k=1}^{N} w(x_k) dx_k,$$

then the k-point correlation functions defined by

$$R_{k,n}(x_1, \dots, x_k) = \frac{n!}{(n-k)!} \int \dots \int p(x_1, \dots, x_n) dx_{k+1} \dots dx_n$$

can be expressed as

$$R_{k,n}(x_1, \dots, x_k) = \det \left(\sqrt{w(x_i) w(x_j)} K_n(x_i, x_j) \right)_{i,j=1}^{n}, \tag{1.1}$$

where, if $p_k(\mu, x) = p_k(x)$ denotes the orthonormal polynomial of degree k with respect to the measure $d\mu(x) = w(x)dx$, the function $K_n(x, y)$ is the so-called Christoffel–Darboux kernel for μ defined as

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x) p_k(y). \tag{1.2}$$

This was originally shown for Gaussian ensembles by Mehta and Gaudin in [20], but later this technique was developed for more general ensembles, see in particular [6, (4.89)] or for example [5,7,8,24]. $K_n(x, y)$ can be expressed in terms of p_n and p_{n-1} as

$$K_n(x, y) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x) p_{n-1}(y) - p_{n-1}(x) p_n(y)}{x - y}, \tag{1.3}$$

where $\gamma_n(\mu) = \gamma_n$ denotes the leading coefficient of $p_n(x)$. This is called the Christoffel–Darboux formula.

Because of (1.1), limits of the type

$$\lim_{n \to \infty} \frac{K_n\left(x_0 + \frac{a}{n}, x_0 + \frac{b}{n}\right)}{K_n(x_0, x_0)}, \quad a, b \in \mathbb{R}, \tag{1.4}$$

which are called universality limits, are playing an especially important role in the study of eigenvalue distributions for random matrices. For measures supported on $[-1, 1]$, a new approach for universality limits was developed by D.S. Lubinsky in the seminal papers [17,15,16]. In [17] it was shown that if μ is a finite Borel measure supported on $[-1, 1]$ which is regular in the sense of Stahl–Totik (see (1.13) below) and absolutely continuous with $d\mu(x) = w(x)dx$ in a neighborhood of $x_0 \in (-1, 1)$, where $w(x)$ is also continuous and strictly positive,