

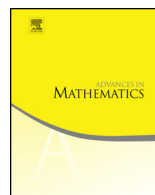


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# Sets of large dimension not containing polynomial configurations

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## ABSTRACT

The main result of this paper is the following. Given countably many multivariate polynomials with rational coefficients and maximum degree  $d$ , we construct a compact set  $E \subset \mathbb{R}^n$  of Hausdorff dimension  $n/d$  which does not contain finite point configurations corresponding to the zero sets of the given polynomials.

Given a set  $E \subset \mathbb{R}^n$ , we study the angles determined by three-point subsets of  $E$ . The main result implies the existence of a compact set in  $\mathbb{R}^n$  of Hausdorff dimension  $n/2$  which does not contain the angle  $\pi/2$ . (This is known to be sharp if  $n$  is even.) We show that there is a compact set of Hausdorff dimension  $n/8$  which does not contain an angle in any given countable set. We also construct a compact set  $E \subset \mathbb{R}^n$  of Hausdorff dimension  $n/6$  for which the set of angles determined by  $E$  is Lebesgue null.

In the other direction, we present a result that every set of sufficiently large dimension contains an angle  $\varepsilon$  close to any given angle.

The main result can also be applied to distance sets. As a corollary we obtain a compact set  $E \subset \mathbb{R}^n$  ( $n \geq 2$ ) of Hausdorff dimension  $n/2$  which does not contain rational distances nor collinear points, for which the distance set

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is Lebesgue null, moreover, every distance and direction is realised only at most once by  $E$ .

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### 1. Introduction

The distance set conjecture asserts that for every analytic set  $E \subset \mathbb{R}^n$  ( $n \geq 2$ ) of Hausdorff dimension larger than  $n/2$ , the set of distances formed by  $E$ ,

$$D(E) = \{|x - y| : x, y \in E\} \subset \mathbb{R},$$

has positive Lebesgue measure. The problem was first studied by Falconer [2]. He showed that there is a compact set  $E \subset \mathbb{R}^n$  of Hausdorff dimension  $n/2$  for which  $D(E)$  is Lebesgue null. In the other direction, he proved that  $\dim_H E > (n + 1)/2$  implies that  $D(E)$  has positive Lebesgue measure. According to the current best results, the same holds if  $\dim_H E > n/2 + 1/3$  (see Wolff [12] and Erdoğan [1]). However, there are sequences of distances  $(d_i)$ ,  $d_i \rightarrow 0$ , and compact sets  $E \subset \mathbb{R}^n$  of dimension  $n$  such that  $d_i \notin D(E)$  for every  $i$ .

There has been recent interest in studying the set of angles  $\mathcal{A}(E)$  formed by three-point subsets of a given  $E \subset \mathbb{R}^n$ , see [6,5,7]. A particularly interesting open question is the following: What minimum dimension of a set  $E \subset \mathbb{R}^n$  guarantees that  $\mathcal{A}(E)$  has positive Lebesgue measure? A simple observation is that results for the distance set conjecture give results for the angle set as well. Therefore, for every analytic set  $E \subset \mathbb{R}^n$  with

$$\dim_H E > \min(n/2 + 4/3, n - 1),$$

$\mathcal{A}(E)$  has positive Lebesgue measure (Theorem 3.6). In the other direction, we construct a compact set  $E \subset \mathbb{R}^n$  of Hausdorff dimension  $\max(n/6, 1)$  such that  $\mathcal{A}(E)$  is Lebesgue null (Theorem 3.5).

The following question was raised by Keleti. How large dimension can a Borel set  $E \subset \mathbb{R}^n$  ( $n \geq 2$ ) have such that  $\mathcal{A}(E)$  does not contain a given angle  $\alpha$ ?

It follows from a theorem of Mattila that  $\dim_H E > n - 1$  implies  $\mathcal{A}(E) = [0, \pi]$ , and that  $\dim_H E > \lceil \frac{n}{2} \rceil$  implies  $\pi/2 \in \mathcal{A}(E)$ , see [6]. It is also known that if  $E$  is an infinite set, then  $E$  contains angles arbitrarily close to 0 and  $\pi$ , see [6]. It is proved in [6], that (independently of  $n$ )  $\dim_H E > 1$  guarantees angles arbitrarily close to  $\pi/2$ . Also, for some absolute constant  $C$ ,  $\dim_H E > C\delta^{-1} \log \delta^{-1}$  implies that  $E$  contains angles in the  $\delta$  neighbourhood of  $\pi/3$  and  $2\pi/3$ .

Harangi [5] proved that there is a self-similar set  $E \subset \mathbb{R}^n$  of dimension  $c_\delta n$  such that all angles formed by  $E$  are in the  $\delta$  neighbourhood of  $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$ , where  $c_\delta$  is a constant independent of  $n$ . He also showed that there exists a compact set  $E \subset \mathbb{R}^n$  with  $\dim_H E = c\sqrt[3]{n}/\log n$  not containing the angle  $\pi/3$  and  $2\pi/3$ .

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