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Sets of large dimension not containing polynomial configurations

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A B S T R A C T

The main result of this paper is the following. Given countably many multivariate polynomials with rational coefficients and maximum degree d, we construct a compact set $E \subset \mathbb{R}^n$ of Hausdorff dimension n/d which does not contain finite point configurations corresponding to the zero sets of the given polynomials.

Given a set $E \subset \mathbb{R}^n$, we study the angles determined by three-point subsets of E. The main result implies the existence of a compact set in $\mathbb{R}^n$ of Hausdorff dimension $n/2$ which does not contain the angle $\pi/2$. (This is known to be sharp if n is even.) We show that there is a compact set of Hausdorff dimension $n/8$ which does not contain an angle in any given countable set. We also construct a compact set $E \subset \mathbb{R}^n$ of Hausdorff dimension $n/6$ for which the set of angles determined by E is Lebesgue null.

In the other direction, we present a result that every set of sufficiently large dimension contains an angle ε close to any given angle.

The main result can also be applied to distance sets. As a corollary we obtain a compact set $E \subset \mathbb{R}^n$ ($n \geq 2$) of Hausdorff dimension $n/2$ which does not contain rational distances nor collinear points, for which the distance set

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is Lebesgue null, moreover, every distance and direction is realised only at most once by E.

1. Introduction

The distance set conjecture asserts that for every analytic set $E \subset \mathbb{R}^n$ $(n \geq 2)$ of Hausdorff dimension larger than $n/2$, the set of distances formed by E,

$$D(E) = \{|x - y| \,:\, x, y \in E\} \subset \mathbb{R},$$

has positive Lebesgue measure. The problem was first studied by Falconer [2]. He showed that there is a compact set $E \subset \mathbb{R}^n$ of Hausdorff dimension $n/2$ for which $D(E)$ is Lebesgue null. In the other direction, he proved that $\dim_H E > (n+1)/2$ implies that $D(E)$ has positive Lebesgue measure. According to the current best results, the same holds if $\dim_H E > n/2 + 1/3$ (see Wolff [12] and Erdoğan [1]). However, there are sequences of distances (d_i), $d_i \to 0$, and compact sets $E \subset \mathbb{R}^n$ of dimension n such that $d_i \notin D(E)$ for every i.

There has been recent interest in studying the set of angles $\mathcal{A}(E)$ formed by three-point subsets of a given $E \subset \mathbb{R}^n$, see [6,5,7]. A particularly interesting open question is the following: What minimum dimension of a set $E \subset \mathbb{R}^n$ guarantees that $\mathcal{A}(E)$ has positive Lebesgue measure? A simple observation is that results for the distance set conjecture give results for the angle set as well. Therefore, for every analytic set $E \subset \mathbb{R}^n$ with

$$\dim_H E > \min(n/2 + 4/3,\, n - 1),$$

$\mathcal{A}(E)$ has positive Lebesgue measure (Theorem 3.6). In the other direction, we construct a compact set $E \subset \mathbb{R}^n$ of Hausdorff dimension $\max(n/6,\, 1)$ such that $\mathcal{A}(E)$ is Lebesgue null (Theorem 3.5).

The following question was raised by Keleti. How large dimension can a Borel set $E \subset \mathbb{R}^n$ $(n \geq 2)$ have such that $\mathcal{A}(E)$ does not contain a given angle α?

It follows from a theorem of Mattila that $\dim_H E > n - 1$ implies $\mathcal{A}(E) = [0, \pi]$, and that $\dim_H E > \lceil \frac{n}{2} \rceil$ implies $\pi/2 \in \mathcal{A}(E)$, see [6]. It is also known that if E is an infinite set, then E contains angles arbitrarily close to 0 and π, see [6]. It is proved in [6], that (independently of n) $\dim_H E > 1$ guarantees angles arbitrarily close to $\pi/2$. Also, for some absolute constant C, $\dim_H E > C\delta^{-1} \log \delta^{-1}$ implies that E contains angles in the δ neighbourhood of $\pi/3$ and $2\pi/3$.

Harangi [5] proved that there is a self-similar set $E \subset \mathbb{R}^n$ of dimension $c_\delta n$ such that all angles formed by E are in the δ neighbourhood of $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$, where c_δ is a constant independent of n. He also showed that there exists a compact set $E \subset \mathbb{R}^n$ with $\dim_H E = c\sqrt[3]{n}/\log n$ not containing the angle $\pi/3$ and $2\pi/3$.