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The structure of the free boundary in the fully nonlinear thin obstacle problem [☆]

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ABSTRACT

We study the regularity of the free boundary in the fully nonlinear thin obstacle problem. Our main result establishes that the free boundary is C^1 near any regular point. This extends to the fully nonlinear setting the celebrated result of Athanasopoulos–Caffarelli–Salsa [1].
The proofs we present here are completely independent from those in [1], and do not rely on any monotonicity formula. Furthermore, an interesting and novel feature of our proofs is that we establish the regularity of the free boundary without classifying blow-ups, a priori they could be non-homogeneous and/or non-unique. We do not classify blow-ups but only prove that they are 1D on $\{x_n = 0\}$.

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1. Introduction

The aim of this paper is to study the regularity of free boundaries in thin obstacle problems.

1.1. Known results

The first regularity results for thin obstacle problems were already established in the seventies by Lewy [21], Frehse [13], Caffarelli [5], and Kinderlehrer [17]. In particular, for the Laplacian Δ, it was proved in [5] that solutions are $C^{1,\alpha}$, for some small $\alpha > 0$.

The regularity of free boundaries, however, was an open problem during almost 30 years. One of the main difficulties in the understanding of free boundaries in thin obstacle problems is that there is not an a priori preferred order at which the solution detaches from the obstacle (blow-ups may have different homogeneities), as explained next.

In the classical (thick) obstacle problem it is not difficult to show that

$$0 < cr^2 \leq \sup_{B_r(x_0)} u \leq Cr^2 \tag{1.1}$$

at all free boundary points x_0, where u is the solution of the problem (after subtracting the obstacle φ). Then, thanks to this, the blow-up sequence $u(x_0 + rx)/r^2$ converges to a global solution u_0, and such solutions u_0 can be shown to be convex and completely classified; see [4,6,7] and also [14,20].

The situation is quite different in thin obstacle problems, in which one does not have (1.1). This was resolved for the first time in Athanasopoulos–Caffarelli–Salsa [1], by using *Almgren's frequency function*. Thanks to this powerful tool, one may take the blow-up sequence

$$\frac{u(x_0 + rx)}{\left(\fint_{\partial B_r(x_0)} u^2\right)^{1/2}},$$

and it converges to a *homogeneous* function u_0 of degree μ, for some $\mu > 1$. Then, by analyzing an eigenvalue problem on S^{n-1}, one can prove that

$$\mu < 2 \qquad \Longrightarrow \qquad \mu = \frac{3}{2},$$

and for $\mu = \frac{3}{2}$ one can completely classify blow-ups. This leads to the optimal $C^{1,\frac{1}{2}}$ regularity of solutions and, using also a boundary Harnack inequality in "slit" domains, to the $C^{1,\alpha}$ regularity of the free boundary near *regular points*—those at which $\mu < 2$.

The main result of [1] may be summarized as follows: if u solves the thin obstacle problem for the Laplacian Δ and with zero obstacle, then for each free boundary point x_0 one has: