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Relative Chow stability and extremal metrics

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ABSTRACT

We prove that the existence of extremal metrics implies asymptotically relative Chow stability. An application of this is the uniqueness, up to automorphisms, of extremal metrics in any polarization.

1. Introduction

On a compact Kähler manifold M, extremal metrics are introduced by Calabi as canonical representations in Kähler classes ([3]). Extremal metrics are critical points of Calabi functional

$$\mathrm{Cal}(\omega) = \int_M S(\omega)^2 \omega^n$$

restricted to a given Kähler class, where $S(\omega)$ is the scalar curvature of ω. Extremal metrics are generalization of constant scalar curvature Kähler (cscK) metrics. There is a deep relationship between the existence of canonical metrics on polarized manifolds and

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the concept of stability. It was conjectured by Yau ([33]), Tian ([32]) and Donaldson ([7]) that the existence of cscK metrics (and more generally extremal metrics) in a polarization is equivalent to the stability of the polarized manifold. The link between the existence and stability is provided by projective embeddings.

Let (M, L) be a polarized manifold. For any $k \gg 0$ using sections of $H^0(M, L^k)$, there exist embeddings of M into complex projective spaces. For any hermitian metric h on L such that $\omega = \sqrt{-1}\,\bar{\partial}\partial \log h$ is a Kähler form on M, one can use L^2-orthonormal bases of $H^0(M, L^k)$ to embed M into complex projective spaces. For any such embedding, the pullback of the Fubini–Study metric to M rescaled by a factor of k^{-1} is a Kähler metric in the class of $2\pi c_1(L)$. In [31], Tian proved that this sequence of rescaled metrics converges to ω. In [6], Donaldson proved that if ω has constant scalar curvature and $\mathrm{Aut}(M, L)/\mathbb{C}^*$ is discrete, then there exists unique "balanced" embedding of M into complex projective spaces using sections of $H^0(M, L^k)$ for $k \gg 0$. These balanced embeddings are zeros of some finite dimensional moment maps and are essentially unique. Moreover by pulling back Fubini–Study metrics to M using these embeddings and rescaling by a factor of k^{-1}, we get a sequence of Kähler metrics in the class of $2\pi c_1(L)$ that converges to the cscK metric. An immediate consequence of Donaldson's theorem is the uniqueness of constant scalar curvature Kähler metrics in the class of $2\pi c_1(L)$ under the discreteness assumption for $\mathrm{Aut}(M, L)/\mathbb{C}^*$.

On the other hand, a result of Zhang ([36]), Luo ([16]), Paul ([24]) and Phong and Sturm ([25]) gives a geometric invariant theory (GIT) interpretation balanced embeddings. They show that the existence of a unique balanced metric on L^k is equivalent to the Chow stability of (M, L^k). Therefore, by Donaldson's theorem, the existence of cscK metrics implies asymptotically Chow stability of (M, L) under the discreteness assumption. Later, Mabuchi showed that under vanishing of some obstructions, one can drop the discreteness assumption ([18,20]). These obstructions appear if the action of the automorphism group of M on the Chow line is non-trivial. In that case, any one parameter subgroup of automorphisms of M that acts nontrivially on the Chow line destabilizes the Chow point. Therefore, the Chow point fails to be semi-stable. So, it is natural to study only those one parameter subgroups that are perpendicular, in some appropriate sense, to the group of automorphisms of M. In analogy to the Kempf–Ness theorem, Székelyhidi introduced the notion of relative stability in [30]. Our main theorem is to prove that the existence of extremal Kähler metrics implies asymptotically relative Chow stability in the sense of [19] and [30]. The main theorem of this article is the following.

Theorem 1.1. *Let (M, L) be a polarized manifold and $T \subset Aut_0(M, L)$ be a maximal torus. Suppose that there exists a T-invariant extremal Kähler metric ω_∞ in the class of $2\pi c_1(L)$. Then there exists a positive integer r only depending on (M, L) and a sequence of T-invariant relatively balanced metrics $\widetilde{\omega}_k$ on (M, L^{rk}) for $k \gg 0$ such that the sequence of rescaled metrics $\omega_k := \frac{1}{rk}\widetilde{\omega}_k$ converges to ω_∞ in C^∞-topology.*

Similar to the case of trivial automorphism group, we have the following.