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A topological transformation group without non-trivial equivariant compactifications

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ABSTRACT

There is a countable metrizable group acting continuously on the space of rationals in such a way that the only equivariant compactification of the space is a singleton. This is obtained by a recursive application of a construction due to Megrelishvili, which is a metric fan equipped with a certain group of homeomorphisms. The question of existence of a topological transformation group with the property in the title was asked by Yu.M. Smirnov in the 1980s.

1. Introduction

Let a (Hausdorff) topological group G act continuously on a (Tychonoff) topological space X. An *equivariant compactification* of X is a compact space K, equipped with a

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continuous action by G, together with a continuous equivariant map $i: X \to K$ having a dense image in K. (See [23–25,22,12,14,16].) The question of interest is whether, given G and X, there exists a compactification into which X embeds topologically, that is, i is a homeomorphism onto its image.

For instance, this is so if a topological group G acts on itself by left translations, in which case the compactification is the greatest ambit of G ([4], [19], p. 42). If the acting group G is discrete, then its action on βX, the Stone–Čech compactification of X, is clearly continuous, so i is the canonical homeomorphic embedding $X \hookrightarrow \beta X$. For non-discrete G, the action on the Stone–Čech compactification βX is continuous just in some exceptional cases. Nevertheless, if the acting group G is locally compact, then every G-space X admits an equivariant compactification into which X embeds homeomorphically, this is an important result by de Vries [24,25]. (For compact Lie groups, the result was known to Palais much earlier, [18].)

For a long time it was unknown whether the same conclusion holds in the case of every acting topological group (the question was advertised by de Vries in 1975, see [23]). However, Megrelishvili [12,13] has shown it is not so, by constructing an example where both G and X are Polish, yet the embedding i is never topological for any equivariant compactification of X. More examples can be found in the work of Megrelishvili and Scarr [15] and Sokolovskaya [21]. (The earliest claim of an example [1] was withdrawn by its authors.) In some examples of Megrelishvili, the mapping i is not even injecive.

Some time in the 1980s, Yuri M. Smirnov asked about the existence of a topological group G acting on a (non-trivial) space X in such a way that the only equivariant compactification of X is the singleton. The question was apparently never mentioned in Smirnov's papers; however, it was well known among the Moscow general and geometric topologists, and later documented in Megrelishvili's papers [13,14]. (See also a discussion in [19], Rem. 3.1.6.) Here we notice that a topological transformation group conjectured by Smirnov indeed exists.

The construction is based on the example of Megrelishvili [12,13], the first ever in which the mapping $i: X \to K$ is not a topological embedding. In this construction, the space X is the metric fan, joining the base point $*$ with countably many endpoints x_n with the help of intervals of unit length, and equipped with the graph metric. The group G is a certain subgroup of the group of homeomorphisms of X with the compact-open topology, chosen in a certain subtle way. Megrelishvili has shown that in every equivariant compactification of X, the images of the points x_n converge to the image of the base point $*$, so the compactification mapping i is not homeomorphic. A further observation by Megrelishvili is that if one joins two copies of such a fan by identifying their respective endpoints and extends the group action over the resulting space, then the image of the sequence of (x_n) will converge to the image of each the two base points, thus these two points have a common image, and consequently i is not even injective.

The idea of our example is, starting with a topological transformation group, to attach to it a copy of the double Megrelishvili fan (in its rational version) at every pair of distinct points x, y. As a result, in any equivariant compactification of the resulting