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A perspective on non-commutative frame theory ☆

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ABSTRACT

This paper extends the fundamental results of frame theory to a non-commutative setting where the role of locales is taken over by étale localic categories. This involves ideas from quantale theory and from semigroup theory, specifically Ehresmann semigroups, restriction semigroups and inverse semigroups. We prove several main results. To start with, we establish a duality between the category of complete restriction monoids and the category of étale localic categories. The relationship between monoids and categories is mediated by a class of quantales called restriction quantal frames. This result builds on the work of Pedro Resende on the connection between pseudogroups and étale localic groupoids but in the process we both generalize and simplify: for example, we do not require involutions and, in addition, we render his result functorial. A wider class of quantales, called multiplicative Ehresmann quantal frames, is put into a correspondence with those localic categories where the multiplication structure map is semiopen, and all the other structure maps are open. We also project down to topological spaces and, as a result, extend the classical adjunction between locales and topological spaces to an adjunction between étale localic categories and étale topological categories. In fact, varying morphisms, we obtain several adjunctions. Just as in the commutative case,

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Boolean algebra
Sober space
Spectral space
Spatial frame

we restrict these adjunctions to spatial-sober and coherent-spectral equivalences. The classical equivalence between coherent frames and distributive lattices is extended to an equivalence between coherent complete restriction monoids and distributive restriction semigroups. Consequently, we deduce several dualities between distributive restriction semigroups and spectral étale topological categories. We also specialize these dualities for the setting where the topological categories are cancellative or are groupoids. Our approach thus links, unifies and extends the approaches taken in the work by Lawson and Lenz and by Resende.

Contents

1. Introduction and preliminaries

1.1. Introduction

The first goal of this paper is to connect, unify and extend the two approaches adopted in [31] and [22,23] in relating inverse semigroups with étale localic or topo-