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Explicit Salem sets in $\mathbb{R}^2$

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ABSTRACT

We construct explicit (i.e., non-random) examples of Salem sets in $\mathbb{R}^2$ of dimension s for every $0 \leq s \leq 2$. In particular, we give the first explicit examples of Salem sets in $\mathbb{R}^2$ of dimension $0 < s < 1$. This extends a theorem of Kaufman.

1. Introduction

1.1. Basic notation

For $x \in \mathbb{R}^d$, $|x| = |x|_\infty = \max_{1 \leq i \leq d} |x_i|$ and $|x|_2 = (\sum_{i=1}^d |x_i|^2)^{1/2}$. For $x, y \in \mathbb{R}^d$, $\langle x, y \rangle = \sum_{i=1}^d x_i y_i$ is the Euclidean inner product. If A is a finite set, $|A|$ is the cardinality

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of A. The expression $a \lesssim b$ stands for "there is a constant $c > 0$ such that $a \le cb$." The expression $a \gtrsim b$ is analogous.

1.2. Background

If μ is a finite Borel measure on $\mathbb{R}^d$, then the Fourier transform of μ is defined by

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle \xi, x \rangle} d\mu(x) \quad \forall \xi \in \mathbb{R}^d.$$

It is a classic result essentially due to Frostman [13] that the Hausdorff dimension of any Borel set $A \subseteq \mathbb{R}^d$ can be expressed as

$$\dim_H A = \sup \left\{ s \in [0,d] : \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^{s-d} d\xi < \infty \text{ for some } \mu \in \mathcal{P}(A) \right\},$$

where $\mathcal{P}(A)$ denotes the set of all Borel probability measures with compact support contained in A.

The Fourier dimension of a set $A \subseteq \mathbb{R}^d$ is defined to be

$$\dim_F A = \sup \left\{ s \in [0,d] : \sup_{0 \neq \xi \in \mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^s < \infty \text{ for some } \mu \in \mathcal{P}(A) \right\}.$$

As general references for Hausdorff and Fourier dimension, see [11,27,28,33]. Recent papers by Ekström, Persson, and Schmeling [9] and Fraser, Orponen, and Sahlsten [12] have revealed some interesting subtleties about Fourier dimension.

Plainly, for every Borel set $A \subseteq \mathbb{R}^d$,

$$\dim_F A \le \dim_H A.$$

Every k-dimensional plane in $\mathbb{R}^d$ with $k < d$ has Fourier dimension 0 and Hausdorff dimension k. The middle-thirds Cantor set in $\mathbb{R}$ has Fourier dimension 0 and Hausdorff dimension $\ln 2 / \ln 3$.

Sets $A \subseteq \mathbb{R}^d$ with

$$\dim_F A = \dim_H A$$

are called Salem sets.

Every ball in $\mathbb{R}^d$ is a Salem set of dimension d. Every countable set in $\mathbb{R}^d$ is a Salem set of dimension zero. Less trivially, every sphere in $\mathbb{R}^d$ is a Salem set of dimension $d-1$. Salem sets in $\mathbb{R}^d$ of dimension $s \neq 0, d-1, d$ are more complicated.