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On dimensions of groups with cocompact classifying spaces for proper actions

Ian J. Leary *, Nansen Petrosyan

School of Mathematical Sciences, University of Southampton, Southampton, SO17 1BJ, United Kingdom

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ABSTRACT

We construct groups G that are virtually torsion-free and have virtual cohomological dimension strictly less than the minimal dimension for any model for $\underline{E}G$, the classifying space for proper actions of G. They are the first examples that have these properties and also admit cocompact models for $\underline{E}G$. We exhibit groups G whose virtual cohomological dimension and Bredon cohomological dimension are two that do not admit any 2-dimensional contractible proper G-CW-complex.

1. Introduction

If G is a virtually torsion-free group, the virtual cohomological dimension vcd$G$ is defined to be the cohomological dimension of a torsion-free finite-index subgroup $H \leq G$; a lemma due to Serre shows that this is well defined [6, VIII.3.1]. Now suppose that X is a contractible G-CW-complex that is *proper*, in the sense that all cell stabilizers

* Corresponding author.
E-mail addresses: i.j.leary@soton.ac.uk (I.J. Leary), n.petrosyan@soton.ac.uk (N. Petrosyan).

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are finite. In this case any torsion-free subgroup H will act freely on X and so X/H is a classifying space or Eilenberg–Mac Lane space BH for H. In particular, vcd$G$ provides a lower bound for the dimension of any such X. K.S. Brown asked whether this lower bound is always attained [5, ch. 2] or [6, VIII.11]:

Brown's question *(Weak form).* Does every virtually torsion-free group G admit a contractible proper G-CW-complex of dimension vcd$G$?

Until now, this form of Brown's question has remained unanswered. We give examples of groups G with vcd$G = 2$ that do not admit any 2-dimensional contractible proper G-CW-complex in Theorem 1.3 below.

One reason why this question has been so elusive is that there are many different equivariant homotopy types of contractible proper G-CW-complexes. The most natural example is the classifying space for proper G-actions, $\underline{E}G$, which plays the same role in the homotopy category of proper G-CW-complexes as EG plays for free G-CW-complexes. A model for $\underline{E}G$ is a proper G-CW complex X such that for any finite $F \leq G$, the F-fixed point set X^F is contractible. Such an X always exists, and is unique up to equivariant homotopy equivalence. Let $\underline{\mathrm{gd}}G$ denote the minimal dimension of any model for $\underline{E}G$.

The version of Brown's question that concerns $\underline{E}G$ [5, ch. 2] or [6, VIII.11] is usually asked in the form:

Brown's question *(Strong form).* Does $\underline{\mathrm{gd}}G = \mathrm{vcd}G$ for every virtually torsion-free G?

We prefer to split this question into two separate questions. There is an algebraic dimension $\underline{\mathrm{cd}}G$ that bears a close relationship to $\underline{\mathrm{gd}}G$, analogous to the relationship between cohomological dimension and the minimal dimension of an Eilenberg–Mac Lane space. It can be shown that $\underline{\mathrm{cd}}G = \underline{\mathrm{gd}}G$ except that there may exist G for which $\underline{\mathrm{cd}}G = 2$ and $\underline{\mathrm{gd}}G = 3$, and $\underline{\mathrm{cd}}G$ is an upper bound for the cohomological dimension of any torsion-free subgroup of G [20]. In view of this we may split the strong form of Brown's question into two parts, one geometric and one algebraic.

Does there exist G for which $\underline{\mathrm{gd}}G \neq \underline{\mathrm{cd}}G$?
Does there exist virtually torsion-free G for which $\underline{\mathrm{cd}}G > \mathrm{vcd}G$?

Examples of virtually torsion-free groups G for which $\underline{\mathrm{gd}}G = 3$ and $\underline{\mathrm{cd}}G = 2$ were given in [2]. These groups G are Coxeter groups. Examples of G for which $\underline{\mathrm{cd}}G > \mathrm{vcd}G$ were given in [17], and more recently in [21,12]. The advantage of the examples in [21,12] is that in some sense they have the least possible torsion. For any virtually torsion-free G, it can be shown that $\underline{\mathrm{cd}}G$ is bounded by the sum $\mathrm{vcd}G + \ell(G)$, where $\ell(G)$ is the maximal length of a chain of non-trivial finite subgroups of G [19, 6.4]. This bound is attained for the examples in [21,12] but not for the examples in [17]. To date, all constructions of groups G for which $\underline{\mathrm{cd}}G > \mathrm{vcd}G$ have used finite extensions of Bestvina–Brady