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Fundamental group functors in descent-exact homological categories

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We study the notion of a fundamental group in the framework of descent-exact homological categories. This setting is sufficiently large to include several categories of "algebraic" nature such as almost abelian categories, semi-abelian categories, and categories of topological semi-abelian algebras. For many adjunctions in this context, we describe the fundamental groups by generalised Brown–Ellis–Hopf formulae for the integral homology of groups.

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1. Introduction

In [13], we pursued the study of the categorical notion of a fundamental group introduced in [35] and provided a generalised version of the Hopf formula for the description of the fundamental group within the semi-abelian context [36]. Examples of semi-abelian categories are the categories of groups, Lie algebras, compact Hausdorff groups, crossed modules, and similar non-abelian structures. In the present work, we define and study higher fundamental groups within the wider context of descent-exact homological categories [3]. This allows us to cover many other categories. For example, let us just mention here the categories of topological groups, locally compact abelian groups, and Banach spaces (with bounded linear maps).

In order to understand what is a descent-exact homological category, let us first recall Tierney's well-known description of abelian categories:

$$\text{abelian} = \text{additive} + \text{Barr-exact}$$

Let us also recall that a category is Barr-exact [1] when it is regular (i.e. it is finitely complete, it has a coequaliser for every kernel pair, and regular epimorphisms are pullback stable) and every internal equivalence is the kernel pair of some morphism. It turns out that in a Barr-exact category $\mathscr{C}$, every regular epimorphism $f\colon E \to B$ is effective for descent, which means that the pullback functor $f^*\colon (\mathscr{C} \downarrow B) \to (\mathscr{C} \downarrow E)$ is monadic. This can be viewed as a form of exactness condition on a category, that we call here descent-exactness (see [25] for a general notion of exactness). Thus, the kind of categories we consider are

$$\underbrace{\text{pointed} + \text{protomodular} + \text{regular}}_{\text{homological}} + \text{descent-exact}$$

where, in presence of the other axioms, the protomodularity [6] condition can be equivalently expressed by saying that the split short five lemma holds. These axioms have numerous consequences. For instance homological lemmas such as the snake lemma or the 3 by 3 lemma still hold in this general context (see the monograph [3] for a general introduction to homological and semi-abelian categories). Note that the descent-exact homological category of topological groups is neither additive nor exact.

The notion of a fundamental group is related to the concept of normal extension coming from categorical Galois theory [32]. In order to give an idea of this relation, let us consider what happens in a simple case. With respect to the reflection of the category Gp of groups into the category Ab of abelian groups given by the abelianisation functor $\mathrm{ab}\colon \mathrm{Gp} \to \mathrm{Ab}$ (we write $\eta\colon 1 \Rightarrow \mathrm{ab}$ for the unit of this reflection), we say that a surjective homomorphism $p\colon E \to B$ (an extension) is a normal extension if the first projection π_1 of its kernel pair

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