Advances in Mathematics 310 (2017) 235–289

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Automorphisms of smooth canonically polarized surfaces in positive characteristic ☆

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ARTICLE INFO

Article history:
Received 31 August 2015
Received in revised form 23 January 2017
Accepted 1 February 2017
Communicated by Ravi Vakil

This paper is dedicated to my wife Afroditi and my children Victoria and Marko

MSC:
primary 14J50, 14DJ29, 14J10
secondary 14D23, 14D22

Keywords:
Canonically polarized surfaces
Automorphisms
Moduli
Vector fields
Positive characteristic

ABSTRACT

This paper investigates the geometry of a smooth canonically polarized surface X defined over an algebraically closed field of characteristic $p > 0$ in the case when the automorphism scheme of X is not smooth. In particular, it is shown that a smooth canonically polarized surface X with $1 \leq K_X^2 \leq 2$ and non-smooth automorphism scheme tends to be uniruled and simply connected and is the purely inseparable quotient of a ruled or rational surface by a rational vector field. Moreover, restrictions on certain numerical invariants of X are obtained in order for $\mathrm{Aut}(X)$ to be smooth.

☆ Part of this paper was written during the author's stay at the Max Planck Institute for Mathematics in Bonn, from June to August 2016.

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[2] The author is supported by a Marie Curie International Outgoing Fellowship, grant No. PIOF-GA-2013-624345.

http://dx.doi.org/10.1016/j.aim.2017.02.002
0001-8708/

1. Introduction

The objective of this paper is to investigate the geometry of a smooth canonically polarized surface X defined over an algebraically closed field k of characteristic $p > 0$ such that the automorphism scheme $\mathrm{Aut}(X)$ of X is not smooth or equivalently that X has nontrivial global vector fields. This is a situation that appears exclusively in positive characteristic simply because every group scheme in characteristic zero is smooth.

The importance of this investigation is twofold. Firstly, the automorphism scheme of a variety is a fundamental invariant of it and is a classical object of study. Secondly, the structure of the automorphism scheme of a canonically polarized surface X is closely related to the moduli of canonically polarized surfaces. In particular, the smoothness of the automorphism scheme is an essential condition in order for the moduli stack of canonically polarized surfaces to be Deligne–Mumford [7], an important property of the stack for the reasons that will be explained in the following paragraph.

The classification up to isomorphism of smooth projective algebraic varieties defined over an algebraically closed field k is one of the fundamental problems of algebraic geometry. In order to deal with this problem, the class of varieties is divided into smaller classes where a reasonable answer is expected to exist, a moduli functor is defined and then it is asked whether it is representable. Unfortunately in general the answer is no because the objects that one wants to parametrize have nontrivial automorphisms. There are two alternatives in order to deal with this complication. Coarse moduli spaces and stacks. Coarse moduli spaces are geometric objects whose closed points are in one-to-one correspondence with the parametrized objects but usually they do not support a universal family. Stacks on the other hand are categorical objects which carry information about families. Loosely speaking, to say that a stack is Deligne–Mumford means that there is a family $\mathcal{X} \to S$ such that for any variety X in the moduli problem, there exist finitely many $s \in S$ such that $\mathcal{X}_s \cong X$, up to étale base change any other family is obtained from it by base change and that for any closed point $s \in S$, the completion $\hat{\mathcal{O}}_{S,s}$ pro-represents the local deformation functor $Def(X_s)$. In some sense this family provides a connection between the local moduli functor (which behaves well) and the global one.

Early on in the theory of moduli of surfaces of general type in characteristic zero, it was realized that the correct objects to parametrize are not the surfaces of general type themselves but their canonical models. It is well known that in characteristic zero the moduli functor of canonically polarized surfaces with fixed Hilbert polynomial has a separated coarse moduli space which is of finite type over the base field k. Moreover, the moduli stack of stable surfaces is a separated, Deligne–Mumford stack of finite type [18,15].

In positive characteristic the moduli functor of canonically polarized surfaces with a fixed Hilbert polynomial still has a separated coarse moduli space of finite type over the base field [14,15]. However, the moduli stack of smooth canonically polarized surfaces is not Deligne–Mumford. The reason for this failure in positive characteristic is the existence of smooth canonically polarized surfaces with non-smooth automorphism scheme,