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Connection coefficients for classical orthogonal polynomials of several variables

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ABSTRACT

Connection coefficients between different orthonormal bases satisfy two discrete orthogonal relations themselves. For classical orthogonal polynomials whose weights are invariant under the action of the symmetric group, connection coefficients between a basis consisting of products of hypergeometric functions and another basis obtained from the first one by applying a permutation are studied. For the Jacobi polynomials on the simplex, it is shown that the connection coefficients can be expressed in terms of Tratnik's multivariable Racah polynomials and their weights. This gives, in particular, a new interpretation of the hidden duality between the variables and the degree indices of the Racah polynomials, which lies at the heart of their bispectral properties. These techniques also lead to explicit formulas for connection coefficients of Hahn and Krawtchouk polynomials of several variables, as well as for orthogonal polynomials on balls and spheres.

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1. Introduction

For a positive measure ρ defined on $\mathbb{R}^d$ that satisfies some mild assumptions, the space $\mathcal{V}_n^d$ of orthogonal polynomials of degree n in d variables with respect to the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)g(x)d\rho(x)$$

has dimension $\dim \mathcal{V}_n^d = \binom{n+d-1}{n}$, where $0 \neq P \in \mathcal{V}_n^d$ if P is a polynomial of degree n and $\langle P, Q\rangle = 0$ for all polynomials Q of degree less than n. The elements of a basis $\{P_\nu : \nu \in \mathbb{N}_0^d, |\nu| = n\}$ for $\mathcal{V}_n^d$ may not be orthogonal among themselves. A basis whose elements are orthogonal to each other is called orthogonal, that is, $\langle P_\nu, P_\mu\rangle = 0$ for $\nu \neq \mu$, and it is called orthonormal if, in addition, $\langle P_\nu, P_\nu\rangle = 1$.

If $\mathbb{P}_n := \{P_\nu : \nu \in \mathbb{N}_0^d, |\nu| = n\}$ and $\mathbb{Q}_n := \{Q_\nu : \nu \in \mathbb{N}_0^d, |\nu| = n\}$ are two orthogonal bases of $\mathcal{V}_n^d$, then we can express one in terms of the other; for example,

$$Q_\nu = \sum_{|\mu|=n} c_{\nu,\mu} P_\mu, \qquad |\nu| = n, \quad c_{\nu,\mu} \in \mathbb{R}. \tag{1.1}$$

We call the coefficients $c_{\nu,\mu}$ *connection coefficients* of $\mathbb{Q}_n$ in terms of $\mathbb{P}_n$. Throughout the paper we adopt the convention of using $\widehat{P}_\nu = P_\nu/||P_\nu||$ to denote the orthonormal polynomial when P_ν is an orthogonal polynomial, and we denote by $\widehat{c}_{\nu,\mu}$ the *normalized* connection coefficients between two orthonormal bases. Thus, (1.1) can be rewritten as

$$\widehat{Q}_\nu = \sum_{|\mu|=n} \widehat{c}_{\nu,\mu} \widehat{P}_\mu, \qquad |\nu| = n.$$

If both $\{P_\mu\}$ and $\{Q_\nu\}$ are orthonormal bases, the matrix $\mathbf{C} := (\widehat{c}_{\nu,\mu})$ is orthogonal ([5, p. 67]) and, therefore, satisfies

$$\sum_{|\omega|=n} \widehat{c}_{\nu,\omega}\widehat{c}_{\mu,\omega} = \delta_{\nu,\mu} \quad \text{and} \quad \sum_{|\omega|=n} \widehat{c}_{\omega,\nu}\widehat{c}_{\omega,\mu} = \delta_{\nu,\mu}. \tag{1.2}$$

We are interested in identifying the connection coefficients for classical orthogonal polynomials of both continuous and discrete variables. Especially interesting is the case when $c_{\nu,\mu}$ can be expressed explicitly in terms of discrete orthogonal polynomials and their weights.

Our starting point is the Jacobi polynomials of two variables that are orthogonal with respect to the weight function $x^\alpha y^\beta (1 - x - y)^\gamma$ on the triangle $T^2 = \{(x, y) : x \geq 0, y \geq 0, 1-x-y \geq 0\}$. For this weight function, one orthonormal basis of $\mathcal{V}_n^2$ consists of polynomials $P_j(x, y) := P_{n-j,j}^{\alpha,\beta,\gamma}(x, y)$ for $0 \leq j \leq n$ given in terms of the classical Jacobi polynomials. Another orthonormal basis consists of $Q_j(x, y) := P_{n-j,j}^{\beta,\gamma,\alpha}(y, 1 - x - y)$,