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Invertibility of sparse non-Hermitian matrices

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ABSTRACT

We consider a class of sparse random matrices of the form $A_n = (\xi_{i,j}\delta_{i,j})_{i,j=1}^n$, where $\{\xi_{i,j}\}$ are i.i.d. centered random variables, and $\{\delta_{i,j}\}$ are i.i.d. Bernoulli random variables taking value 1 with probability p_n, and prove a quantitative estimate on the smallest singular value for $p_n = \Omega(\frac{\log n}{n})$, under a suitable assumption on the spectral norm of the matrices. This establishes the invertibility of a large class of sparse matrices. For $p_n = \Omega(n^{-\alpha})$ with some $\alpha \in (0,1)$, we deduce that the condition number of A_n is of order n with probability tending to one under the optimal moment assumption on $\{\xi_{i,j}\}$. This in particular, extends a conjecture of von Neumann about the condition number to sparse random matrices with heavy-tailed entries. In the case that the random variables $\{\xi_{i,j}\}$ are i.i.d. sub-Gaussian, we further show that a sparse random matrix is singular with probability at most $\exp(-cnp_n)$ whenever p_n is above the critical threshold $p_n = \Omega(\frac{\log n}{n})$. The results also extend to the case when $\{\xi_{i,j}\}$ have a non-zero mean. We further find quantitative estimates on the smallest singular value of the adjacency matrix of a directed Erdős–Réyni graph whenever

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its edge connectivity probability is above the critical threshold $\Omega(\frac{\log n}{n})$.

1. Introduction

This paper establishes the bounds on the condition number of a sparse random matrix with independent identically distributed (i.i.d.) entries and on the probability that such matrix is singular.

For a $n \times n$ real matrix A_n its *singular values* $s_k(A_n)$, $k = 1, 2, \ldots, n$, are the eigenvalues of $|A_n| = \sqrt{A_n^* A_n}$ arranged in non-increasing order. The maximum and the minimum singular values are often of particular interest, and they can be defined as

$$s_{\max}(A_n) := s_1(A_n) := \sup_{x \in S^{n-1}} \|A_n x\|_2, \quad s_{\min}(A_n) := s_n(A_n) := \inf_{x \in S^{n-1}} \|A_n x\|_2,$$

where $S^{n-1} := \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ and $\|\cdot\|_2$ denotes the Euclidean norm of a vector. This definition means that the largest singular value $s_{\max}(A_n)$ is the operator or spectral norm of the matrix A_n, and the smallest singular value $s_{\min}(A_n)$ provides a *quantitative measure* of the invertibility of A_n:

$$s_{\min}(A_n) = \inf \{\|A_n - B\| : \det(B) = 0\},$$

where $\|A_n - B\|$ denotes the operator norm of the $n \times n$ matrix $A_n - B$. Another such measure is the *condition number* defined as

$$\sigma(A_n) := \frac{s_{\max}(A_n)}{s_{\min}(A_n)},$$

which often serves a measure of stability of matrix algorithms in numerical linear algebra.

In this paper we obtain lower bounds on the smallest singular value of a class of *sparse* random matrices, and then finding appropriate upper bounds on the maximum singular value, we deduce that the condition number of such matrices is well controlled, and therefore they are well invertible (see Theorem 1.1, Corollary 1.5 and Corollary 1.8).

Another class of random matrices which are of interest in combinatorics and graph theory are the adjacency matrices of random graphs. Graphs, more precisely, their edges can be either undirected or directed. Both directed and undirected graphs are abundant in real life. One of the simplest, and widely studied models in the undirected random graph literature is the Erdős–Réyni random graph. Here we consider the directed version of that model (see Definition 1.10), and show that the smallest singular value and condition number of the adjacency matrix of such random graphs are well controlled (see Theorem 1.11).