Advances in Mathematics 310 (2017) 557–609

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Cartesian modules over representations of small categories

Sergio Estrada [1], Simone Virili [*,2]

Departamento de Matemáticas, Universidad de Murcia, Campus Espinardo 30100, Murcia, Spain

ARTICLE INFO

Article history:
Received 4 June 2015
Received in revised form 17 January 2017
Accepted 26 January 2017
Communicated by Henning Krause

Dedicated to Luigi Salce on the occasion of his retirement.

MSC:
primary 18E05, 18G15, 18F20, 18A25

Keywords:
Preadditive category
Cartesian module
Representation
Pure derived category

ABSTRACT

We introduce the new concept of cartesian module over a pseudofunctor R from a small category to the category of small preadditive categories. Already the case when R is a (strict) functor taking values in the category of commutative rings is sufficient to cover the classical construction of quasi-coherent sheaves of modules over a scheme. On the other hand, our general setting allows for a good theory of contravariant additive locally flat functors, providing a geometrically meaningful extension of a classical Representation Theorem of Makkai and Paré. As an application, we relate and extend some previous constructions of the pure derived category of a scheme.

* Corresponding author.
E-mail addresses: sestrada@um.es (S. Estrada), virili.simone@gmail.com (S. Virili).

[1] The first named author was supported by the research grant 18394/JLI/13 of the Fundación Séneca - Agencia de Ciencia y Tecnología de la Región de Murcia in the framework of III PCTRM 2011–2014. Furthermore, he is grateful to the Department of Mathematics of the M.I.T. for its hospitality.

[2] The second named author was partially supported by Fondazione Cassa di Risparmio di Padova e Rovigo (Progetto di Eccellenza "Algebraic structures and their applications"), and by the projects DGI MINECO MTM2011-28992-C02-01 and MINECO MTM2014-53644-P (Spain).

http://dx.doi.org/10.1016/j.aim.2017.01.030

Contents

1. Introduction

Let X be a small site (that is, a small category whose Grothendieck topology is defined by a pretopology, see [31]). The usual way of defining a ringed site is by considering pairs $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is a sheaf of commutative rings. More generally a ringed category $(X, \mathcal{O}_X)$ is a pair such that X is a small category and $\mathcal{O}_X$ is a presheaf of commutative rings on X. Then, the category of presheaves of $\mathcal{O}_X$-modules on X can be defined. On the other hand, a (not necessarily commutative) ring may be regarded as a special case of a small preadditive category, that is, a small category R such that $R(a,b)$ is an Abelian group, for each $a, b \in \mathrm{Ob}R$, and morphism composition distributes over addition. So the category of modules over a preadditive category naturally arises. There are many sources in the literature which deal with this generalization. Quoting from [25], "[...] there have been several papers concerned with replacing theorems about rings by theorems about additive[3] categories. What does not seem to be generally realized is the

[3] In [25], Mitchell uses the term "additive" for what is usually known as "preadditive".