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Surfaces of minimal degree of tame representation type and mutations of Cohen–Macaulay modules ☆

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We provide two examples of smooth projective surfaces of tame CM type, by showing that the parameter space of isomorphism classes of indecomposable ACM bundles with fixed rank and determinant on a rational quartic scroll in $\mathbb{P}^5$ is either a single point or a projective line. These turn out to be the only smooth projective ACM varieties of tame CM type besides elliptic curves, [1].

For surfaces of minimal degree and wild CM type, we classify rigid Ulrich bundles as Fibonacci extensions. For $\mathbb{F}_0$ and $\mathbb{F}_1$, embedded as quintic or sextic scrolls, a complete classification of rigid ACM bundles is given.

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Introduction

Let $X \subset \mathbb{P}^n$ be a smooth positive-dimensional closed subvariety over an algebraically closed field $\Bbbk$, and assume that the graded coordinate ring $\Bbbk[X]$ of X is Cohen–Macaulay, i.e. X is ACM (arithmetically Cohen–Macaulay). Then X supports infinitely many indecomposable ACM sheaves $\mathscr{E}$ (i.e. whose $\Bbbk[X]$-module of global sections $\mathrm{H}^0_*(X, \mathscr{E})$ is a maximal Cohen–Macaulay), unless X is $\mathbb{P}^n$ itself, or a quadric hypersurface, or a rational normal curve, or one of the two sporadic cases: the Veronese surface in $\mathbb{P}^5$ and (cf. §1.2) the rational cubic scroll $S(1, 2)$ in $\mathbb{P}^4$, see [12].

Actually, for most ACM varieties X, much more is true. Namely X supports families of arbitrarily large dimension of indecomposable ACM bundles, all non-isomorphic to one another (varieties like this are of "geometrically of wild CM type" or simply "CM-wild"). CM-wild varieties include curves of genus ≥ 2, hypersurfaces of degree $d \geq 4$ in $\mathbb{P}^n$ with $n \geq 2$, complete intersections in $\mathbb{P}^n$ of codimension ≥ 3, having one defining polynomial of degree ≥ 3 (cf. [9,10]), the third Veronese embedding of any variety of dimension ≥ 2 cf. [24]. In many cases, these families are provided by Ulrich bundles, i.e. those $\mathscr{E}$ such that $\mathrm{H}^0_*(X, \mathscr{E})$ achieves the maximum number of generators, namely $d_X \operatorname{rk}(\mathscr{E})$, where we write d_X for the degree of X. For instance, Segre embeddings are treated in [8], smooth rational ACM surfaces in $\mathbb{P}^4$ in [25], cubic surfaces and threefolds in [5,6], del Pezzo surfaces in [27,7].

In spite of this, there is a special class of varieties X with intermediate behaviour, namely X supports continuous families of indecomposable ACM bundles, all non-isomorphic to one another, but, for each rank r, these bundles form finitely (or countably) many irreducible families of dimension at most one. Then X is called of tame CM type, or of tame representation type. It is the case of the elliptic curve, [1].

In this note we provide the first examples of smooth positive-dimensional projective CM-tame varieties, besides elliptic curves. Part of this was announced in [15].

Theorem A. *Let X be a smooth surface of degree 4 in $\mathbb{P}^5$. Then, for any $r \geq 1$, there is a family of isomorphism classes of indecomposable Ulrich bundles of rank $2r$, parametrized by $\mathbb{P}^1$. Conversely, any indecomposable ACM bundle on X is rigid or belongs to one of these families (up to a twist). In particular, X is of tame CM type.*

Recalling the classification by del Pezzo and Bertini of smooth varieties of *minimal degree*, i.e. with $d_X = \operatorname{codim}(X) + 1$, as the Veronese surface in $\mathbb{P}^5$ and rational normal scrolls (cf. [11]), we see two things. On one hand, both CM-finite varieties and our examples have minimal degree, actually a surface of degree 4 in $\mathbb{P}^5$ is a quartic scroll. Incidentally, these have the same graded Betti numbers as the Veronese surface in $\mathbb{P}^5$, which is CM-finite. On the other hand the remaining varieties of minimal degree are CM-wild by [23]; in fact in [23] also quartic scrolls are claimed to be of wild CM type: the gap in the argument only overlooks our two examples, cf. Remark 2.2. By the following result, these examples complete the list of non-CM-wild varieties in a broad sense.