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Tukey classification of some ideals on ω and the lattices of weakly compact sets in Banach spaces ☆

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A B S T R A C T

We study the lattice structure of the family of weakly compact subsets of the unit ball B_X of a separable Banach space X, equipped with the inclusion relation (this structure is denoted by $\mathcal{K}(B_X)$) and also with the parametrized family of "almost inclusion" relations $K \subseteq L+\varepsilon B_X$, where $\varepsilon > 0$ (this structure is denoted by $\mathcal{AK}(B_X)$). Tukey equivalence between partially ordered sets and a suitable extension to deal with $\mathcal{AK}(B_X)$ are used. Assuming the axiom of analytic determinacy, we prove that separable Banach spaces fall into four categories, namely: $\mathcal{K}(B_X)$ is equivalent either to a singleton, or to ω^ω, or to the family $\mathcal{K}(\mathbb{Q})$ of compact subsets of the rational numbers, or to the family $[\mathfrak{c}]^{<\omega}$ of all finite subsets of the continuum. Also under the axiom of analytic determinacy, a similar classification of $\mathcal{AK}(B_X)$ is obtained. For separable Banach spaces not containing ℓ^1, we prove in ZFC that $\mathcal{K}(B_X) \sim \mathcal{AK}(B_X)$ are equivalent to either $\{0\}$, ω^ω, $\mathcal{K}(\mathbb{Q})$

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or $[\mathfrak{c}]^{<\omega}$. The lattice structure of the family of all weakly null subsequences of an unconditional basis is also studied.

1. Introduction

The purpose of this paper is to establish a classification of separable Banach spaces according to how complicated the lattice of weakly compact subsets is. Let $\mathcal{K}(B_X)$ denote the family of all weakly compact subsets of the unit ball B_X of a Banach space X, that we view as a partially ordered set endowed with inclusion. The way in which we measure the complexity of $\mathcal{K}(B_X)$ is through Tukey reduction. This has become a standard way to compare partially ordered sets, proven useful to isolate some essential features of the ordered structure [36]. Let us recall that two upwards-directed partially ordered sets are Tukey equivalent if and only if they are order isomorphic to cofinal subsets of some third upwards-directed partially ordered set. Our first main result is the following:

Theorem A *($\mathbf{\Sigma^1_1 D}$). If X is a separable Banach space, then $\mathcal{K}(B_X)$ is Tukey equivalent to one of the following partially ordered sets:*

(i) *either to a singleton,*
(ii) *or to ω^ω (ordered pointwise),*
(iii) *or to the family $\mathcal{K}(\mathbb{Q})$ of compact subsets of the rational numbers (ordered by inclusion),*
(iv) *or to the family $[\mathfrak{c}]^{<\omega}$ of all finite subsets of the continuum (ordered by inclusion).*

The symbol ($\mathbf{\Sigma^1_1 D}$) in this and later results means that the statement holds under the axiom of analytic determinacy (which is consistent with ZFC if one believes in large cardinals). A reader unfamiliar with determinacy axioms can think that, in practical terms, Theorem A holds for any *reasonable* Banach space, not arising from any set-theoretic oddity. The case (i) corresponds to reflexivity, so the result can be interpreted as saying that non-reflexive separable Banach spaces split into three categories, depending on three canonical patterns of disposition in the lattice of weakly compact sets. When X^* (the dual of X) is separable (for the norm topology), Theorem A holds in ZFC without any determinacy axiom required, and (iv) never happens. This particular case is a corollary to a result of Fremlin [18], who established the Tukey classification of the lattices of compact subsets of coanalytic metric spaces. Our main contribution is therefore the case of non-separable dual. In the case of separable Banach spaces not containing ℓ^1, the classification of Theorem A corresponds to the following well-studied classes of spaces:

(i) reflexive spaces,
(ii) non-reflexive spaces with separable dual and the PCP (point of continuity property),