Advances in Mathematics 310 (2017) 993–1016

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Heisenberg uniqueness pairs for some algebraic curves in the plane

Deb Kumar Giri, R.K. Srivastava *

Department of Mathematics, Indian Institute of Technology, Guwahati, 781039, India

A R T I C L E I N F O

Article history:
Received 2 July 2015
Received in revised form 27 October 2016
Accepted 9 February 2017
Communicated by Charles Fefferman

MSC:
primary 42A38
secondary 44A35

Keywords:
Bessel function
Convolution
Fourier transform

A B S T R A C T

A Heisenberg uniqueness pair is a pair (Γ, Λ), where Γ is a curve and Λ is a set in $\mathbb{R}^2$ such that whenever a finite Borel measure μ having support on Γ which is absolutely continuous with respect to the arc length on Γ satisfies $\hat{\mu}|_\Lambda = 0$, then it is identically 0. In this article, we investigate the Heisenberg uniqueness pairs corresponding to the spiral, hyperbola, circle and certain exponential curves. Further, we work out a characterization of the Heisenberg uniqueness pairs corresponding to four parallel lines. In the latter case, we observe a phenomenon of interlacing of three trigonometric polynomials.

1. Introduction

The concept of the Heisenberg uniqueness pair has been first introduced in an influential article by Hedenmalm and Montes-Rodríguez (see [7]). We would like to mention that Heisenberg uniqueness pair up to a certain extent is similar to an annihilating pair of Borel measurable sets of positive measure as described by Havin and Jöricke [6]. Fur-

* Corresponding author.
E-mail addresses: deb.giri@iitg.ernet.in (D.K. Giri), rksri@iitg.ernet.in (R.K. Srivastava).

http://dx.doi.org/10.1016/j.aim.2017.02.019

ther, the notion of Heisenberg uniqueness pair has a sharp contrast to the known results about determining sets for measures by Sitaram et al. [2,15], due to the fact that the determining set Λ for the function $\hat{\mu}$ has also been considered a thin set.

In addition, the question of determining the Heisenberg uniqueness pair for a class of finite measures has also a significant similarity with the celebrated result due to M. Benedicks (see [3]). That is, support of a function $f \in L^1(\mathbb{R}^n)$ and its Fourier transform $\hat{f}$ both cannot be of finite measure simultaneously. Later, various analogues of the Benedicks theorem have been investigated in different set ups, including the Heisenberg group and Euclidean motion groups (see [12,14,16]).

In particular, if Γ is compact, then $\hat{\mu}$ is a real analytic function having exponential growth and it can vanish on a very delicate set. Hence in this case, finding the Heisenberg uniqueness pairs becomes little easier. However, this question becomes immensely difficult for the measure supported on a non-compact curve. Eventually, the Heisenberg uniqueness pair is a natural invariant to the theme of the well studies uncertainty principle for the Fourier transform.

In the article [7], Hedenmalm and Montes-Rodríguez have shown that the pair (hyperbola, some discrete set) is a Heisenberg uniqueness pair. As a dual problem, a weak* dense subspace of $L^\infty(\mathbb{R})$ has been constructed to solve the Klein–Gordon equation. Further, a complete characterization of the Heisenberg uniqueness pairs corresponding to any two parallel lines has been given by Hedenmalm and Montes-Rodríguez (see [7]).

Afterward, a considerable amount of work has been done pertaining to the Heisenberg uniqueness pair, in the plane as well as in the higher dimensional Euclidean spaces.

Recently, N. Lev [10] and P. Sjölin [17] have independently shown that circle and certain system of lines are HUP corresponding to the unit circle S^1. Further, F.J. Gonzalez Vieli [19] has generalized HUP corresponding to circle in the higher dimension and shown that a sphere whose radius does not lie in the zero set of the Bessel functions $J_{(n+2k-2)/2};\ k \in \mathbb{Z}_+$, the set of non-negative integers, is a HUP corresponding to the unit sphere S^{n-1}.

P. Sjölin [18] has investigated some of the Heisenberg uniqueness pairs corresponding to the parabola. Subsequently, D. Blasi Babot [1] has given a characterization of the Heisenberg uniqueness pairs corresponding to a certain system of three parallel lines. However, an exact analogue for the finitely many parallel lines is still open.

In a major development, P. Jaming and K. Kellay [8] have given a unifying proof for some of the Heisenberg uniqueness pairs corresponding to the hyperbola, polygon, ellipse and graph of the functions $\varphi(t) = |t|^\alpha$, whenever $\alpha > 0$ through dynamical system approach.

Let Γ be a finite disjoint union of smooth curves in $\mathbb{R}^2$. Let $X(\Gamma)$ be the space of all finite complex-valued Borel measure μ in $\mathbb{R}^2$ which is supported on Γ and absolutely continuous with respect to the arc length measure on Γ. For $(\xi, \eta) \in \mathbb{R}^2$, the Fourier transform of μ is defined by