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The spectral estimates for the Neumann–Laplace operator in space domains

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In this paper we prove discreteness of the spectrum of the Neumann–Laplacian (the free membrane problem) in a large class of non-convex space domains. The lower estimates of the first non-trivial Neumann eigenvalue are obtained in terms of geometric characteristics of Sobolev mappings. The suggested approach is based on Sobolev–Poincaré inequalities that are obtained with the help of a geometric theory of composition operators on Sobolev spaces. These composition operators are induced by generalizations of conformal mappings that are called as mappings of bounded 2-distortion (weak 2-quasiconformal mappings).

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1. Introduction

The classical upper estimate for the first nontrivial Neumann eigenvalue of the Laplace operator

$$\mu_1(\Omega) \leq \mu_1(\Omega^*) = \frac{p_{n/2}^2}{R_*^2}$$

was proved by Szegö [38] for simply connected planar domains via a conformal mappings technique ("the method of conformal normalization") and by Weinberger [44] for domains in $\mathbb{R}^n$. In this inequality $p_{n/2}$ denotes the first positive zero of the function $(t^{1-n/2}J_{n/2}(t))'$, and Ω^* is an n-ball of the same n-volume as Ω with R_* as its radius. In particular, if $n = 2$ we have $p_1 = j'_{1,1} \approx 1.84118$ where $j'_{1,1}$ denotes the first positive zero of the derivative of the Bessel function J_1.

More detailed upper estimates for planar domains were obtained in [33] and [25] via "the method of conformal normalization". The upper estimates of the Laplace eigenvalues with the help of different techniques were intensively studied in the recent decades, see, for example, [1–3,11,27].

Situation with lower estimates is more complicated. The classical result by Payne and Weinberger [32] states that in convex domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$

$$\mu_1(\Omega) \geq \frac{\pi^2}{d(\Omega)^2},$$

where $d(\Omega)$ is a diameter of a convex domain Ω. Unfortunately in non-convex domains $\mu_1(\Omega)$ can not be estimated in the terms of Euclidean diameters. It can be seen by considering a domain consisting of two identical squares connected by a thin corridor [6]. In [7,8] lower estimates involved the isoperimetric constant relative to Ω were obtained.

In the works [20,21] we returned to a conformal mappings techniques and obtained lower estimates of $\mu_1(\Omega)$ in the terms of the hyperbolic (conformal) radius of Ω for a large class of general (non-necessary convex) domains $\Omega \subset \mathbb{R}^2$. For example, this class includes some domains with fractal boundaries which Hausdorff dimension can be any number of the half interval $[1, 2)$.

Our method is different from "the method of conformal normalization" and based on the variational formulation of spectral problems and on the geometric theory of composition operators on Sobolev spaces, developed in our previous papers [14,39,40,42,43]. Roughly speaking we "transferred" known estimates (from convex Lipschitz domains) to "general" domains with a help of composition operators induced by conformal mappings.

The variational formulation of the spectral problem for the Laplace operator is usually based on the Dirichlet (energy) integral

$$\|u \mid L_2^1(\Omega)\|^2 = \int\limits_{\Omega} |\nabla u(x)|^2 \, dx,$$

and was established in [34] by Lord Rayleigh.